

# CLT for large dimensional general Fisher matrices and its applications in high-dimensional data analysis

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## Abstract

Random Fisher matrices arise naturally in multivariate statistical analysis and understanding the properties of its eigenvalues is of primary importance for many hypothesis testing problems like testing the equality between two multivariate population covariance matrices, or testing the independence between sub-groups of a multivariate random vector. This paper is concerned with the properties of a large-dimensional Fisher matrix when the dimension of the population is proportionally large compared to the sample size. Most of existing works on Fisher matrices deal with a particular Fisher matrix where populations have i.i.d components so that the population covariance matrices are all identity. In this paper, we consider general Fisher matrices with arbitrary population covariance matrices. The first main result of the paper establishes the limiting distribution of the eigenvalues of a Fisher matrix while in a second main result, we provide a central limit theorem for a wide class of functionals of its eigenvalues. Some applications of these results are also proposed for testing hypotheses on high-dimensional covariance matrices.

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## 1 Introduction

For testing the equality of variances from two populations, a well-known statistic is the Fisher statistic defined as the ratio of two sample variances. Its multivariate counter-part is a random *Fisher matrix* defined by

$$\mathbf{F} := \mathbf{B}_1 \mathbf{B}_2^{-1} \quad (1.1)$$

where the  $\mathbf{B}_j$ 's are sample covariance matrices from two independent samples, say  $\{\boldsymbol{\xi}_k, 1 \leq k \leq n_1\}$  and  $\{\boldsymbol{\eta}_\ell, 1 \leq \ell \leq n_2\}$  with population covariance matrices  $\boldsymbol{\Sigma}_1$  and  $\boldsymbol{\Sigma}_2$ , respectively.

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Of primary importance are the so-called *linear spectral statistics* (LSS) of the matrix  $\mathbf{F}$  of form

$$W_{\mathbf{n}} = \sum_{i=1}^p f(\lambda_i^{\mathbf{F}}), \quad (1.2)$$

where  $\lambda_i^{\mathbf{F}}$ s are the eigenvalues of  $\mathbf{F}$  with the notation  $\mathbf{n} = (n_1, n_2)$ . Fisher matrices, especially its eigenvalues, arise in many hypothesis testing problems in multivariate analysis. Examples include the test of the equality hypothesis  $\Sigma_1 = \Sigma_2$  where the likelihood ratio (LR) statistic is simplified to a functional of eigenvalues of a Fisher matrix, see Bai et al. [7]. In multivariate analysis of variance (MANOVA), the test on the equality of means is reduced to a statistic depending on a Fisher matrix which is a function of the “between” sum of squares and the “within” sum of squares (Anderson [1, p. 346]). In multivariate linear regression, the likelihood ratio criterion for testing linear hypotheses about regression coefficients is expressed as a function of the eigenvalues of a Fisher matrix (Anderson [1, p. 298]). To test the independence between sub-groups of a multivariate population, the LR statistic is a functional of a Fisher matrix defined by sub-matrices of sample covariance matrices (Anderson [1, p. 381]). Fisher matrices appear also in the canonical correlation analysis, see Yang and Pan [25] for a recent account.

This paper concerns the high-dimensional situation where the population dimension  $p$  is large compared to the sample sizes  $n_1$  and  $n_2$ . It is now well understood that classical procedures as those presented in Anderson [1] become impracticable or dramatically lose efficiency with high-dimensional data. For example, the deficiency of the Hotelling’s  $T^2$  statistic has been reported in Dempster [11] and Bai and Saranadasa [2]. Regarding hypothesis testing on high-dimensional covariance matrices, many recent works appeared in the literature, see e.g. [7], [10], [13], [21], [22], [23], and [24]. However, most of these works concern the one-sample situation and in those treating the two-sample situation (except [7]), the test statistics are often proposed through an ad-hoc distance measure so that they do not involve the corresponding Fisher matrices. Indeed, as it can be seen from the multivariate analysis examples discussed earlier, Fisher matrices and its eigenvalues appear naturally in procedures based on the Gaussian likelihood functions.

In the literature from random matrix theory and assuming that the dimension grows to infinity proportionally to sample sizes, the convergence of the eigenvalues of a Fisher matrix to a limiting distribution has been studied by several authors, see e.g. [3], [8], [16], [17], [18], and [26]. As for central limit theorems for linear spectral statistics, Chatterjee [9] establishes the existence of a Gaussian limit assuming that the populations are Gaussian. However, his method doesn’t provide explicit formula for the asymptotic mean and asymptotic covariances of the Gaussian limit. A closely related piece of work is that of

Bai and Silverstein [5] which establishes a CLT for spectral statistics of a general sample covariance matrices of form  $\mathbf{B}_1 \mathbf{T}_p$  where  $\mathbf{B}_1$  is a sample covariance matrix and  $\mathbf{T}_p$  is a non-random Hermitian matrix. This CLT is later refined in [15] where the original restriction on the values of the fourth moments of the population components is removed. However, the CLT in [5] cannot cover spectral statistics of a Fisher matrix by replacing  $\mathbf{T}_p$  by  $\mathbf{B}_2^{-1}$  for the reason that the centering term of this CLT would become a random term without an explicit expression. To overcome this difficulty, Zheng [27] establishes a CLT for spectral statistics of a Fisher matrix which has a non-random and explicit centering term. In particular, the components of the observations  $\boldsymbol{\xi}_i$  and  $\boldsymbol{\eta}_j$  can have arbitrary values of the fourth moment. To our best knowledge, this is the only CLT reported in the literature for spectral statistics of Fisher matrix. However, this CLT has a severe limitation in that it is assumed that the population covariance matrices are equal i.e.  $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ . Although the derivation of this CLT is complex and highly non trivial, it has a small impact on the statistical problems mentioned above where the population covariance matrices  $\boldsymbol{\Sigma}_i$  can be arbitrary and not necessarily equal. Specifically for the test of the equality hypothesis, “ $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ ” and assuming that the population are Gaussian, this CLT enables us to find the distribution of the LR statistic under the null hypothesis, but not under any alternative hypothesis, that is, the size of the test can be found by this CLT and not the power function.

The main contribution of the paper is the establishment of the central limit theorem for linear spectral statistics  $\{W_n\}$  of a general Fisher matrix where the population covariance matrices  $\boldsymbol{\Sigma}_i$  are arbitrary. Under this scheme and as a preparatory step, we also establish a limiting distribution for its eigenvalues and give an explicit equation satisfied by its Stieltjes transform. Due to the fact that the population covariance matrices are arbitrary, the establishment of these results have required several new techniques compared to the existing literature on the central limit theory although the general scheme remains similar to the one used in [5, 27]. A significantly different tool used here is another CLT reported in [28] for random matrix of type  $\mathbf{S}^{-1}T$  where  $\mathbf{S}$  is a standard sample covariance matrix (with i.i.d. standardised components) and  $T$  a nonnegative definite and deterministic Hermitian matrix. These two papers are related each other but focus on different random matrices.

The paper is organized as follows. In Section 2 we first introduce the asymptotic scheme and the technical assumptions used, and then establishes the limiting spectral distribution of the eigenvalues. Section 3 presents the CLT for linear spectral statistics of general Fisher matrices which is the main result of the paper. Section 4 gives two algorithms to approximate the limiting spectral density, mean function and covariance function in CLT for linear spectral statistics. In Section 5, we discuss some applications of the results to

hypothesis testing and confidence intervals about high-dimensional covariance matrices. Technical lemmas and proofs are postponed to Appendix A.

## 2 Limiting spectral distribution of large dimensional general $F$ -matrices

Following Bai and Silverstein [5] and Zheng [27], we will impose the following structure on the observation model. Assume that the samples can be expressed as

$$\boldsymbol{\xi}_k = \boldsymbol{\Sigma}_1^{1/2} \mathbf{X}_{\cdot k}, \quad 1 \leq k \leq n_1; \quad \boldsymbol{\eta}_\ell = \boldsymbol{\Sigma}_2^{1/2} \mathbf{Y}_{\cdot \ell}, \quad 1 \leq \ell \leq n_2;$$

where the observations matrices

$$\begin{aligned} \mathbf{X} &:= (\mathbf{X}_{\cdot 1}, \dots, \mathbf{X}_{\cdot n_1}) = (X_{jk} : 1 \leq j \leq p, 1 \leq k \leq n_1), \\ \mathbf{Y} &:= (\mathbf{Y}_{\cdot 1}, \dots, \mathbf{Y}_{\cdot n_2}) = (Y_{j\ell} : 1 \leq j \leq p, 1 \leq \ell \leq n_2), \end{aligned}$$

are the upper-left corners, of size  $p \times n_1$  and  $p \times n_2$ , of two independent arrays of independent random variables  $\{X_{jk}, j, k = 1, 2, \dots\}$  and  $\{Y_{j\ell}, j, \ell = 1, 2, \dots\}$ , respectively. The corresponding sample covariance matrices become

$$\mathbf{B}_1 = \frac{1}{n_1} \sum_{k=1}^{n_1} \boldsymbol{\xi} \boldsymbol{\xi}^* = \boldsymbol{\Sigma}_1^{1/2} \mathbf{S}_1 (\boldsymbol{\Sigma}_1^{1/2})^*, \quad \text{with } \mathbf{S}_1 = \frac{1}{n_1} \sum_{k=1}^{n_1} \mathbf{X}_{\cdot k} \mathbf{X}_{\cdot k}^*, \quad (2.1)$$

$$\mathbf{B}_2 = \frac{1}{n_2} \sum_{\ell=1}^{n_2} \boldsymbol{\eta} \boldsymbol{\eta}^* = \boldsymbol{\Sigma}_2^{1/2} \mathbf{S}_2 (\boldsymbol{\Sigma}_2^{1/2})^*, \quad \text{with } \mathbf{S}_2 = \frac{1}{n_2} \sum_{\ell=1}^{n_2} \mathbf{Y}_{\cdot \ell} \mathbf{Y}_{\cdot \ell}^*. \quad (2.2)$$

Because  $\mathbf{F} = \mathbf{B}_1 \mathbf{B}_2^{-1}$  has the same eigenvalues as  $\mathbf{S}_1 (\mathbf{T}_p^{1/2})^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2}$  where  $\mathbf{T}_p^{1/2} = \boldsymbol{\Sigma}_2^{-1/2} \boldsymbol{\Sigma}_1^{1/2}$ , we can define as well the Fisher matrix to be  $\mathbf{F} := \mathbf{S}_1 (\mathbf{T}_p^{1/2})^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2}$ . It is also noticed that obviously, the matrix  $\mathbf{S}_2$  should be invertible (almost surely) so that in our asymptotic analysis, we will assume  $n_2 > p$  for large  $p$  and  $n_2$ .

Throughout the paper, *empirical spectral distribution* (or ESD) of square matrix refers to the empirical distribution generated by its eigenvalues. We consider the following assumptions.

**Assumption [A]** The two double arrays  $\{X_{ki}, i, k = 1, 2, \dots\}$  and  $\{Y_{ki}, i, k = 1, 2, \dots\}$  consist of independent but not necessarily identically distributed random variables with mean 0 and variance 1.

**Assumption [B1]** For any fixed  $\eta > 0$  and when  $n_1, n_2, p \rightarrow \infty$ ,

$$\frac{1}{n_1 p} \sum_{j=1}^p \sum_{k=1}^{n_1} \mathbb{E} [|X_{jk}|^2 I_{\{|X_{jk}| \geq \eta \sqrt{n_1}\}}] \rightarrow 0, \quad \frac{1}{n_2 p} \sum_{j=1}^p \sum_{k=1}^{n_2} \mathbb{E} [|Y_{jk}|^2 I_{\{|Y_{jk}| \geq \eta \sqrt{n_2}\}}] \rightarrow 0. \quad (2.3)$$

**Assumption [B2]** The two arrays are either both real, we then set the indicator  $\kappa = 2$ ; or both complex, we then set  $\kappa = 1$ , with homogeneous 4th moments:  $\mathbb{E}|X_{jk}|^4 = 1 + \kappa + \beta_x + o(1)$ ,  $\mathbb{E}|Y_{jk}|^4 = 1 + \kappa + \beta_y + o(1)$ . Moreover, for any fixed  $\eta > 0$  when  $n_1, n_2, p \rightarrow \infty$ ,

$$\frac{1}{n_1 p} \sum_{j=1}^p \sum_{k=1}^{n_1} \mathbb{E} [|X_{jk}|^4 I_{\{|X_{jk}| \geq \eta \sqrt{n_1}\}}] \rightarrow 0, \quad \frac{1}{n_2 p} \sum_{j=1}^p \sum_{k=1}^{n_2} \mathbb{E} [|Y_{jk}|^4 I_{\{|Y_{jk}| \geq \eta \sqrt{n_2}\}}] \rightarrow 0. \quad (2.4)$$

In addition,  $\mathbb{E} X_{jk}^2 = o(n_1^{-1})$ ,  $\mathbb{E} Y_{jk}^2 = o(n_2^{-1})$  when both arrays  $\{X_{jk}\}$  and  $\{Y_{jk}\}$  are complex.

**Assumption [C]** The sample sizes  $n_1, n_2$  and the dimension  $p$  grow to infinity in such a way that

$$y_{n_1} := p/n_1 \rightarrow y_1 \in (0, +\infty), \quad y_{n_2} := p/n_2 \rightarrow y_2 \in (0, 1). \quad (2.5)$$

**Assumption [D]** The matrices  $\mathbf{T}_p$  are non-random and nonnegative definite Hermitian matrices and the sequence  $\{\mathbf{T}_p\}$  is bounded in spectral norm. Moreover, the ESD  $H_p$  of  $\mathbf{T}_p$  tends to a proper nonrandom probability measure  $H$  when  $p \rightarrow \infty$ .

The assumptions (2.3) and (2.4) are standard Lindeberg type conditions which are necessary for the existence of the limiting spectral distribution for  $\mathbf{F}$ , and for the CLT for LSS of  $\mathbf{F}$ , respectively. Moreover, under these conditions, the variables  $X_{ik}$  and  $Y_{ik}$ 's can be truncated at size  $\eta_p \sqrt{p}$  ( $\eta_p \downarrow 0$ ) without altering asymptotic results.

The following notations are used throughout the paper:

$$\mathbf{n} = (n_1, n_2), \quad \mathbf{y}_{\mathbf{n}} = (y_{n_1}, y_{n_2}), \quad \mathbf{y} = (y_1, y_2), \quad h^2 = y_1 + y_2 - y_1 y_2.$$

In the sequel, the limiting results will be investigated under the regime (2.5) that will be simply referred as  $\mathbf{n} \rightarrow \infty$ . Some useful concepts are now recalled. The Stieltjes transform of a positive Borel measure  $G$  on the real line is defined by

$$m_G(z) \equiv \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ = \{z : z \in \mathbb{C}, \Im(z) > 0\}. \quad (2.6)$$

Table 1: Notations for distributions and Stieltjes transforms (S.T.) of random matrices

Matrix	ESD / S.T.	LSD / S.T.
$\mathbf{F} = \mathbf{S}_1 \{\mathbf{T}_p^{1/2}\}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2}$	$U_{\mathbf{n}} / m_{\mathbf{n}}$	$U_{\mathbf{y}} / m_{\mathbf{y}}$
$\mathbf{X}^* \{\mathbf{T}_p^{1/2}\}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2} \mathbf{X}$	$\underline{U}_{\mathbf{n}} / \underline{m}_{\mathbf{n}}$	$\underline{U}_{\mathbf{y}} / \underline{m}_{\mathbf{y}}$
$\{\mathbf{T}_p^{1/2}\}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2}$	$G_{n_2} /$	$G_{y_2} /$

This transform has a natural extension to the lower-half plane by the formula

$$m_G(z) = \overline{m}_G(\bar{z}), \quad \text{for } z \in \mathbb{C}^- = \{z : z \in \mathbb{C}, \Im(z) < 0\}.$$

In addition to  $\mathbf{F}$ , we will also need several other matrices. Table 1 contains the notations used in the sequel for characteristics of these matrices: ESD, LSD and the associated Stieltjes transforms.

The matrices  $\mathbf{F}$  and  $\mathbf{X}^* (\mathbf{T}_p^{1/2})^* \mathbf{S}_2^{-1} \mathbf{T}_p^{1/2} \mathbf{X}$  are companion matrices each other sharing same non null eigenvalues so that we have

$$\underline{m}_{\mathbf{n}}(z) = -\frac{1 - y_{n_1}}{z} + y_{n_1} m_{\mathbf{n}}(z), \quad (2.7)$$

$$\underline{m}_{\mathbf{y}}(z) = -\frac{1 - y_1}{z} + y_1 m_{\mathbf{y}}(z). \quad (2.8)$$

Furthermore, when  $\Sigma_1 = \Sigma_2$ , i.e.,  $\mathbf{T}_p = \mathbf{I}_p$ , it is well-known that the LSD  $U_{\mathbf{y}}$  of  $\mathbf{F}$  and its Stieltjes transform  $m_{\mathbf{y}}(z)$  can be found on Page 79 of Bai and Silverstein [6]. As a first result of the paper, we prove the existence of  $U_{\mathbf{y}}$  and one of its characteristics for general Fisher matrix  $\mathbf{F}$  where  $\mathbf{T}_p$  is a Hermitian matrix.

**Theorem 2.1** *Under Assumptions [A], [B1], [C] and [D],*

- (i) *The matrix  $\mathbf{S}_2^{-1} \mathbf{T}_p$  has a non-random LSD  $G_{y_2}$ . Moreover,  $G_{y_2}$  is characterised by the fact that the transform*

$$m_{y_2}(z) = \int_0^\infty \frac{t}{1 - tz} dG_{y_2}(t),$$

where

$$\underline{m}_{y_2}(z) = -\frac{1 - y_2}{z} + y_2 m_{y_2}(z)$$

is the unique solution to the equation

$$z = -\frac{1}{\underline{m}_{y_2}(z)} + y_2 \int \frac{dH(t)}{t + \underline{m}_{y_2}(z)}, \quad z \in \mathbb{C}^+. \quad (2.9)$$

(ii) The Fisher matrix  $\mathbf{F} = \mathbf{S}_1(\mathbf{T}_p^{1/2})^* \mathbf{S}_2^{-1}(\mathbf{T}_p^{1/2})$  has a non-random LSD  $U_{\mathbf{y}}$ . Moreover,  $U_{\mathbf{y}}$  is characterised by the fact that the Stieltjes transform  $\underline{m}_{\mathbf{y}}(z)$  of its companion measure  $\underline{U}_{\mathbf{y}}$  is the unique solution to the equation

$$z = \frac{h^2 m_0(z)}{y_2(-1 + y_2 \int \frac{m_0(z) dH(t)}{t + m_0(z)})} + \frac{y_1}{y_2} m_0(z), \quad z \in \mathbb{C}^+, \quad (2.10)$$

where  $m_0(z) = \underline{m}_{y_2}(-\underline{m}_{\mathbf{y}}(z))$ .

The proof of this theorem is given in Appendix A.1.

**Remark 2.1** For a given  $z \in \mathbb{C}^+$ , the equation (2.9) has a unique solution  $m_0$  such that  $\Im(m_0) < 0$ . Then, the Stieltjes transform  $\underline{m}(z)$  can be computed by substituting  $z = -\underline{m}_{\mathbf{y}}$  into Equation (2.9), i.e.

$$-\underline{m}_{\mathbf{y}}(z) = -\frac{1}{m_0(z)} + y_2 \int \frac{dH(t)}{t + m_0(z)}. \quad (2.11)$$

In fact by Silverstein [19],  $\underline{m}_{\mathbf{y}}(z)$  is the unique solution to the equation

$$z = -\frac{1}{\underline{m}_{\mathbf{y}}(z)} + y_1 \int \frac{x dG_{y_2}(x)}{1 + x \underline{m}_{\mathbf{y}}(z)}. \quad (2.12)$$

In the sequel, for brevity, the notations  $m_{\mathbf{y}}(z)$  and  $\underline{m}_{\mathbf{y}}(z)$  will be simplified to  $m(z)$  and  $\underline{m}(z)$ , or even to  $m$  and  $\underline{m}$ , respectively, if no confusion would be possible. We will use the notations  $G_{y_{n_2}}$  that are obtained by substituting  $y_{n_2} = p/n_2$  for  $y_2$  in  $G_{y_2}$ .

### 3 CLT for LSS of large dimensional general Fisher matrices

As explained in Introduction, we consider *linear spectral statistics* (LSS) of  $\mathbf{F}$

$$W_{\mathbf{n}} = p \cdot U_{\mathbf{n}}(f) = \sum_{j=1}^p f(\lambda_j^{\mathbf{F}}), \quad (3.1)$$

where  $f$  is an analytic function and  $\{\lambda_j^{\mathbf{F}}\}$  are the eigenvalues of  $\mathbf{F}$ . More precisely, we consider a centered version

$$p[U_{\mathbf{n}}(f) - U_{\mathbf{y}_{\mathbf{n}}}(f)] . \quad (3.2)$$

where  $U_{\mathbf{y}_{\mathbf{n}}}(f) = \int f(x) dU_{\mathbf{y}_{\mathbf{n}}}(x)$ ,  $U_{\mathbf{y}}(x)$  is the LSD of the Fisher matrix and  $U_{\mathbf{y}_{\mathbf{n}}}(x)$  is obtained by substituting  $\mathbf{y}_{\mathbf{n}} = (y_{n_1}, y_{n_2})$  for  $\mathbf{y} = (y_1, y_2)$  in  $U_{\mathbf{y}}(x)$ . Due to the exact

separation theorem (see Bai and Silverstein [4]), for large enough  $n_j$  and  $p$ , the possible point mass at the origin of  $U_{\mathbf{n}}$  will coincide exactly with that of  $U_{\mathbf{y}_{\mathbf{n}}}$ . Therefore, we can restrict the integral (3.2) to their continuous components on  $(0, \infty)$ , i.e.

$$p[U_{\mathbf{n}}(f) - U_{\mathbf{y}_{\mathbf{n}}}(f)] = \sum_{j=1}^p f(\lambda_j^{\mathbf{F}}) I_{(\lambda_j^{\mathbf{F}} > 0)} - p \int f(x) u_{\mathbf{y}_{\mathbf{n}}}(x) dx \quad (3.3)$$

where  $u_{\mathbf{y}_{\mathbf{n}}}(x)$  is the density on  $(0, \infty)$  of  $U_{\mathbf{y}_{\mathbf{n}}}(x)$ .

Regarding the central limit theory on linear spectral statistics of random matrices, it has been well-known ([5, 15, 27]) that the mean and covariance parameters of the limiting Gaussian distribution depend on the values of the fourth moments of the initial variables. When these moments match the Gaussian case, i.e.  $\beta_x = 0$  or  $\beta_y = 0$  in Assumption [B2], the limiting parameters have a simpler expression. Otherwise, they have a more involved expression that depend on other limiting functional of sample covariance matrices. More specifically, if  $\beta_x \neq 0$ , we will need the existence of the following limits

$$\begin{aligned} & \frac{1}{p} \sum_{i=1}^p \mathbb{E} \left[ \mathbf{e}'_i (\mathbf{T}_p^{\frac{1}{2}})^* \mathbf{S}_2^{-\frac{1}{2}} \mathbf{D}_1^{-1} \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i \right. \\ & \quad \times \left. \mathbf{e}'_i (\mathbf{T}_p^{\frac{1}{2}})^* \mathbf{S}_2^{-\frac{1}{2}} \mathbf{D}_1^{-1} \left( \underline{m}(z) \{ \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{\frac{1}{2}} + \mathbf{I}_p \right)^{-1} \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i \right] \longrightarrow h_{m1}(z), \end{aligned} \quad (3.4)$$

$$\begin{aligned} & \frac{1}{n_1 p} \sum_{j=1}^{n_1} \sum_{i=1}^p \mathbf{e}'_i (\mathbf{T}_p^{\frac{1}{2}})^* \mathbf{S}_2^{-\frac{1}{2}} [\mathbf{E}_j \mathbf{D}_j^{-1}(z_1)] \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i \\ & \quad \times \mathbf{e}'_i (\mathbf{T}_p^{\frac{1}{2}})^* \mathbf{S}_2^{-\frac{1}{2}} [\mathbf{E}_j \mathbf{D}_j^{-1}(z_2)] \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i \xrightarrow{i.p.} h_{v1}(z_1, z_2), \end{aligned} \quad (3.5)$$

and if  $\beta_y \neq 0$ , we will need the existence of the limits

$$\begin{aligned} & \frac{1}{p} \sum_{i=1}^p \mathbb{E} \mathbf{e}'_i \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_2 \right)^{-1} \mathbf{e}_i \cdot \mathbf{e}'_i \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_2 \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p + \frac{1}{z} \underline{m}_{y2} \left( \frac{1}{z} \right) \mathbf{I} \right)^{-1} \mathbf{e}_i \\ & \quad \longrightarrow h_M(z), \end{aligned} \quad (3.6)$$

$$\frac{1}{n_2 p} \sum_{j=1}^{n_2} \sum_{i=1}^p \mathbf{e}'_i E_j \left( \frac{1}{z_1} \mathbf{T}_p - \mathbf{S}_{2,j} \right)^{-1} \mathbf{e}_i \cdot \mathbf{e}'_i E_j \left( \frac{1}{z_2} \mathbf{T}_p - \mathbf{S}_{2,j} \right)^{-1} \mathbf{e}_i \xrightarrow{i.p.} h(z_1, z_2). \quad (3.7)$$

Here

$$\mathbf{S}_{2,j} = \mathbf{S}_2 - \frac{1}{n_2} \mathbf{Y}_{\cdot j} \mathbf{Y}_{\cdot j}^*, \quad \mathbf{D}_j(z) = (\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}}) \left( \mathbf{S}_1 - \frac{1}{n_1} \mathbf{X}_{\cdot j} \mathbf{X}_{\cdot j}^* \right) (\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}})^* - z \cdot \mathbf{I}_p,$$

and  $\mathbf{e}_i$  denotes the  $i$ -th vector of the canonical basis of  $\mathbb{C}^p$ .

The following CLT is the main result of the paper.



**Theorem 3.1** *Under the Assumptions [A], [B2], [C] and [D], assume that the limits (3.4)-(3.5) exist whenever  $\beta_x \neq 0$ , and the limits (3.6)-(3.7) exist whenever  $\beta_y \neq 0$ . Let  $f_1, \dots, f_s$  be  $s$  functions analytic in an open domain of the complex plane that enclosed the support interval  $[c_1, c_2]$  of the continuous component of the LSD  $U_{\mathbf{y}}$ . Then, as  $\mathbf{n} \rightarrow \infty$ , the random vector*

$$\{p[U_{\mathbf{n}}(f_j) - U_{\mathbf{y}_{\mathbf{n}}}(f_j)] \ , \quad 1 \leq j \leq s\} \ ,$$

*converges weakly to a Gaussian vector  $(X_{f_1}, \dots, X_{f_s})$  with mean function*

$$\begin{aligned} EX_f &= \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z) \, d \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - y_2 \int \frac{m_0(z)}{t+m_0(z)} dH(t)\right)^2}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} dH(t)} \right) \\ &\quad - \frac{\beta_x y_1}{2\pi i} \cdot \oint_{\mathcal{C}} \frac{z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - y_2 \int \frac{y_2 m_0(z)}{t+m_0(z)} dH(t)\right)^2}{1 - y_2 \int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2} dH(t)}} dz \\ &\quad + \frac{\kappa - 1}{4\pi i} \oint_{\mathcal{C}} f(z) \, d \log \left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2} \right) \\ &\quad + \frac{\beta_y y_2}{2\pi i} \cdot \oint \underline{m}'(z) \frac{\underline{m}^3(z) m_0^3(z) h_M(-\frac{1}{\underline{m}(z)})}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} dH(t)} dz \ , \end{aligned} \quad (3.8)$$

*and covariance function*

$$\begin{aligned} &\text{Cov}(X_{f_i}, X_{f_j}) \\ &= -\frac{\beta_x y_1}{4\pi^2} \cdot \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \partial z_2} dz_1 dz_2 \\ &\quad - \frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{f_i(z_1) f_j(z_2)}{(m_0(z_1) - m_0(z_2))^2} dm_0(z_1) dm_0(z_2) \\ &\quad - \frac{\beta_y y_2}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{\underline{m}'(z_1) \underline{m}'(z_2)}{\underline{m}^2(z_1) \underline{m}^2(z_2)} \frac{\partial^2 \left[ \underline{m}(z_1) m_0(z_1) \underline{m}(z_2) m_0(z_2) h \left( -\frac{1}{\underline{m}(z_1)}, -\frac{1}{\underline{m}(z_2)} \right) \right]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} dz_1 dz_2 \end{aligned} \quad (3.9)$$

*where the contours  $\mathcal{C}$ ,  $\mathcal{C}_1$  and  $\mathcal{C}_2$  all enclose the support of  $U_{\mathbf{y}}$ , and  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are disjoint.*

Similar to CLT's developed in [5, 27], all the limiting parameters depend on contour integrals using the associated Stieltjes transforms. Some specific examples of calculations of such integrals can be found in these references.

We next develop an important special example where the matrices  $\{\mathbf{T}_p\}$  are diagonal. In this case, we find explicit expressions for the limiting functions  $h_M(z)$  and  $h(z_1, z_2)$ . This in turn simplifies the expressions of limiting mean and covariance functions in the CLT.

**Proposition 3.1** *In addition to the assumptions of Theorem 3.1, assume that the matrices  $\mathbf{T}_p$ 's are diagonal. Then, the limits (3.6) and (3.7) exist and equal to*

$$h_M(z) = \int \frac{t}{\left(\frac{t}{z} + \frac{1}{z} \underline{m}_{y_2}\left(\frac{1}{z}\right)\right)^3} dH(t) , \quad (3.10)$$

$$h(z_1, z_2) = \int \frac{1}{\left(\frac{t}{z_1} + \frac{1}{z_1} \underline{m}_{y_2}\left(\frac{1}{z_1}\right)\right) \left(\frac{t}{z_2} + \frac{1}{z_2} \underline{m}_{y_2}\left(\frac{1}{z_2}\right)\right)} dH(t) . \quad (3.11)$$

Consequently, the same conclusions as in Theorem 3.1 hold where the last term of  $EX_f$  in (3.8) is simplified to

$$\frac{\beta_y}{4\pi i} \oint_{\mathcal{C}} f(z) d \left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t + m_0(z))^2} \right) \quad (3.12)$$

and the last term of  $\text{Cov}(X_{f_i}, X_{f_j})$  in (3.9) is simplified to

$$-\frac{\beta_y y_2}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \left[ \int \frac{t^2 dH(t)}{(t + m_0(z_1))^2 (t + m_0(z_2))^2} \right] dm_0(z_1) dm_0(z_2). \quad (3.13)$$

where each of the contours  $\mathcal{C}$ ,  $\mathcal{C}_1$  and  $\mathcal{C}_2$  encloses the support of  $U_{\mathbf{y}}$  and  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are disjoint.

**Remark 3.1** *The contours in Theorem 3.1 and Proposition 3.1 are taken in the  $z$  space. In this case, the contours can be arbitrary provided that they enclose the support of the LSD  $U_{\mathbf{y}}$ . Since the integrands are functions of  $m_0$ , thus the integrals can be taken in the  $m_0$  space using the change of variable  $z \mapsto m_0(z)$ .*

**Remark 3.2** *When  $\mathbf{T}_p$  is an identity matrix, (3.12) and (3.13) are the same as (3.6) and (3.7) in Zheng (2013). That is, Theorem 3.2 in Zheng (2012) is a special case of Theorem 3.1 in this paper when  $\mathbf{T}_p = \mathbf{I}_p$ .*

## 4 Evaluation of the asymptotic parameters $EX_f$ , $\text{Cov}(X_{f_i}, X_{f_j})$ and the limiting density $u_{\mathbf{y}}(x)$

The practical application of Theorem 3.1 or Proposition 3.1 requires to know the limiting spectral density  $u_{\mathbf{y}}(x)$ , the asymptotic mean  $EX_f$  and covariance function  $\text{Cov}(X_{f_i}, X_{f_j})$ . In particular, the last two functions depend on some non trivial contour integrals. In the simple case where  $\mathbf{T}_p = \mathbf{I}_p$  and for simple functions like  $f(x) = x^j$  (monomials) or  $f(x) = \log(x)$ , analytical results can be found exactly, see [27]. However, this is a **very**

particular case and for general population matrices or more complex functions  $f$ , such exact results are not available. In this section, we introduce some numerical procedures to approximate these asymptotic parameters while deliberately placing ourselves in the context of practical application with real data sets. In such a situation, the sample sizes and dimension of data  $(n_1, n_2, p)$  are given and the empirical spectral distribution  $H_p$  of  $\mathbf{T}_p = \Sigma_1 \Sigma_2^{-1}$  is known. In this section, we denote the eigenvalues of  $\mathbf{T}_p$  simply by  $\{\lambda_j^0\}$  so that  $H_p(t) = \frac{1}{p} \sum_{j=1}^p I_{(\lambda_j^0 \leq t)}$ . However in such a concrete application situation, the LSD  $H$  is never known and we need an estimate of  $H$ . A very reasonable and widely used estimate of  $H$  is indeed just  $H_p$ . Here we assume a more general estimate of the form

$$\hat{H}(t) = \sum_{j=1}^p w_j I(\lambda_j^0 \leq t). \quad (4.1)$$

where  $\{w_j\}$  is a family of mixing weights, i.e.  $w_j \geq 0$  and  $\sum w_j = 1$ . This form includes  $H_p$  and many other interesting estimators of  $H$ , e.g. a kernel estimate.

Notice that the parameters  $u_{\mathbf{y}}(x)$ ,  $EX_f$  and  $\text{Cov}(X_{f_i}, X_{f_j})$  all depend on the Stieltjes transform  $m_0(z)$ . We first approximate this transform.

**Lemma 4.1** *Let  $z = x_z + y_z \mathbf{i}$  and  $m_0(z) = u_0 + v_0 \mathbf{i}$  with corresponding real and imaginary parts. We have*

$$x_z = - \frac{h^2 u_0 \left( 1 - y_2 + y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right) - h^2 v_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2}}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(1 + \lambda_j^0 u_0)^2 + (\lambda_j^0)^2 v_0^2} \right)^2} + \frac{y_1 u_0}{y_2}, \quad (4.2)$$

and

$$y_z = - \frac{h^2 u_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} + h^2 v_0 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2} + \frac{y_1 v_0}{y_2}. \quad (4.3)$$

The proof of Lemma 4.1 is given in Appendix. Therefore given  $z = x_z + y_z \mathbf{i}$ ,  $(u_0, v_0)$  are solutions of the nonlinear equations (4.2) and (4.3). These equations can be easily solved using standard computing software to get numerically the values of  $(u_0, v_0)$ , i.e. of  $m_0(z)$ .

Next, the limiting spectral density  $u_{\mathbf{y}}(x)$  can be approximated as indicated below.

**Remark 4.1** By (2.11) and (4.1) of this paper and Theorem B.10 of Bai and Silverstein [6], we have

$$\underline{m}(z) = \frac{1}{m_0(z)} - y_2 \int \frac{dH(t)}{t + m_0(z)} \approx \frac{1}{m_0(z)} - y_2 \sum_{j=1}^p \frac{w_j}{\lambda_j^0 + m_0(z)} \quad (4.4)$$

and

$$u_{\mathbf{y}}(x) = \frac{1}{\pi y_1} \lim_{\varepsilon \rightarrow 0_+} \Im(\underline{m}(x + \varepsilon \mathbf{i})). \quad (4.5)$$

**Remark 4.2** The limiting functions  $h_{m1}(z)$  and  $h_{v1}(z_1, z_2)$  can be approximated as follows

$$\hat{h}_{m1}(z) = \frac{1}{n_1 p} \sum_{j=1}^{n_1} \sum_{i=1}^p B_{1ij}(z) B_{2ij}(z), \quad \hat{h}_{v1}(z_1, z_2) = \frac{1}{n_1 p} \sum_{j=1}^{n_1} \sum_{i=1}^p B_{1ij}(z_1) B_{1ij}(z_2) \quad (4.6)$$

where

$$B_{1ij}(z) = \mathbf{e}_i' \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{D}_j^{-1}(z) \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i$$

and

$$B_{2ij}(z) = \mathbf{e}_i' \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{D}_j^{-1}(z) \left( \underline{m}(z) \{ \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{\frac{1}{2}} + \mathbf{I}_p \right)^{-1} \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \mathbf{e}_i$$

with  $\mathbf{D}_j(z) = (\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}}) \left( \mathbf{S}_1 - \frac{1}{n_1} \mathbf{X}_{\cdot j} \mathbf{X}_{\cdot j}^* \right) (\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}})^* - z \cdot \mathbf{I}_p$ .

The following remark will give a simplified form of the asymptotic mean function  $\mathbb{E}X_f$  and asymptotic covariance function  $\text{Cov}(X_{f_i}, X_{f_j})$ .

**Remark 4.3** In Theorem 3.1<sup>2</sup>, the mean and covariance functions have alternate expressions

$$\begin{aligned} \mathbb{E}X_f &= -\frac{\kappa-1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - y_2 \int \frac{m_0(z)}{t+m_0(z)} dH(t)\right)^2}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} dH(t)} \right) dz \\ &\quad - \frac{\beta_x y_1}{2\pi i} \oint_{\mathcal{C}} f(z) \cdot \frac{z^2 \underline{m}^3(z) h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0(z)}{t+m_0(z)} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2(z)}{(t+m_0(z))^2} dH(t)}} dz \\ &\quad - \frac{\kappa-1}{4\pi i} \oint_{\mathcal{C}} f'(z) \log \left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2} \right) dz \\ &\quad + \frac{\beta_y y_2}{4\pi i} \oint_{\mathcal{C}} f'(z) \left( \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2} \right) dz \end{aligned} \quad (4.7)$$

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<sup>2</sup>To be generalized to the case of Theorem 3.1.

and covariance functions

$$\begin{aligned}
& \text{Cov}(X_{f_i}, X_{f_j}) \\
&= -\frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f'_i(z_1) f'_j(z_2) \log(m_0(z_1) - m_0(z_2)) dz_1 dz_2 \\
&\quad - \frac{\beta_y y_2}{4\pi^2} \oint \oint f'_i(z_1) f'_j(z_2) \left[ \int \frac{t^2 dH(t)}{(t + m_0(z_1))(t + m_0(z_2))} \right] dz_1 dz_2 \\
&\quad - \frac{\beta_x y_1}{4\pi^2} \oint \oint f'_i(z_1) f'_j(z_2) \cdot [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) \cdot h_{v1}(z_1, z_2)] dz_1 dz_2. \tag{4.8}
\end{aligned}$$

Combining the methods devised in Lemma 4.1 and Remark 4.1-4.3, we now describe the general procedure to approximate the limiting spectral density  $u_{\mathbf{y}}(x)$ , the asymptotic mean and covariance functions.

**Algorithm 1: approximating the limiting spectral density  $u_{\mathbf{y}}(x)$**

Cut the support set  $[c_1, c_2]$  of the LSD of Fisher matrix  $\mathbf{F}$  into a mesh set as

$$\mathcal{A} = \left\{ z_j = x_j + \varepsilon \mathbf{i}, x_j = c_1 + \frac{(c_2 - c_1)j}{m}, \quad j = 0, \dots, m \right\},$$

where  $\varepsilon$  is a small step size, e.g.  $10^{-3}$ . By (4.2) and (4.3), we obtain  $m_0(z_j)$  with  $z_j \in \mathcal{A}$ . By (4.4), we obtain  $\underline{m}(z_j)$  with  $z_j \in \mathcal{A}$ . Then by (4.5) let

$$u_{\mathbf{y}}(x_j) \simeq \frac{1}{\pi y_1} \Im(\underline{m}(z_j)) \tag{4.9}$$

we obtain an approximation of the density  $u_{\mathbf{y}}(x_j)$ .

**Algorithm 2: approximating the asymptotic mean function (4.7) and covariance function (4.8)**

*Step 1.* Chose two disjoint contours  $\mathcal{C}_1$  and  $\mathcal{C}_2$  both enclosing the support  $[c_1, c_2]$  of  $u_{\mathbf{y}}$  as depicted on Figure ?? where  $\varepsilon$  and  $\zeta$  are small numbers, e.g.  $\varepsilon = \zeta = 10^{-3}$ .

*Step 2.* Let  $m_1, m_2$  be large integers, e.g.  $10^3$ . Then  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are cut into a grid set as

$$\begin{aligned}
\mathcal{A}_1 = & \left\{ z_k = c_1 - \varepsilon + \left( \zeta - \frac{2\zeta k}{m_1} \right) \mathbf{i}, \quad k = 0, \dots, m_1 \right. \\
& z_{m_1+j} = c_1 - \varepsilon + \frac{(c_2 - c_1 + 2\varepsilon)j}{m_2} - \zeta \mathbf{i}, \quad j = 0, \dots, m_2 \\
& z_{m_1+m_2+k} = c_2 + \varepsilon + \left( -\zeta + \frac{2\zeta k}{m_1} \right) \mathbf{i}, \quad k = 0, \dots, m_1 \\
& \left. z_{2m_1+m_2+j} = c_2 + \varepsilon - \frac{(c_2 - c_1 + 2\varepsilon)j}{m_2} + \zeta \mathbf{i}, \quad j = 0, \dots, m_2 \right\},
\end{aligned}$$

$$\mathcal{A}_2 = \left\{ \begin{aligned} z_k &= c_1 - \frac{\varepsilon}{2} + \left( \frac{\zeta}{2} - \frac{\zeta k}{m_1} \right) \mathbf{i}, k = 0, \dots, m_1 \\ z_{m_1+j} &= c_1 - \frac{\varepsilon}{2} + \frac{(c_2 - c_1 + \varepsilon)j}{m_2} - \frac{\zeta}{2} \mathbf{i}, j = 0, \dots, m_2 \\ z_{m_1+m_2+k} &= c_2 + \frac{\varepsilon}{2} + \left( -\frac{\zeta}{2} + \frac{\zeta k}{m_1} \right) \mathbf{i}, k = 0, \dots, m_1 \\ z_{2m_1+m_2+j} &= c_2 + \frac{\varepsilon}{2} - \frac{(c_2 - c_1 + \varepsilon)j}{m_2} + \frac{\zeta}{2} \mathbf{i}, j = 0, \dots, m_2 \end{aligned} \right\}.$$

*Step 3.* By (4.2) and (4.3), we obtain  $m_0(z_j)$ . By (4.4), we obtain  $\underline{m}(z_j)$ . Then mean function and covariance function are approximated by

$$\begin{aligned} EX_f &\approx -\frac{\kappa-1}{4\pi} \sum_{j=0}^{2m_1+2m_2+3} \Im \left[ f'(z_j) \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left( 1 - y_2 \int \frac{m_0(z_j)}{t+m_0(z_j)} dH(t) \right)^2}{1 - y_2 \int \frac{m_0^2(z_j)}{(t+m_0(z_j))^2} dH(t)} \right) (z_{j+1} - z_j) \right] \\ &\quad - \frac{\beta_x y_1}{2\pi} \sum_{j=0}^{2m_1+2m_2+3} \Im \left[ f(z_j) \cdot \frac{z^2 \underline{m}^3(z_j) h_{m_1}(z_j)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left( 1 - \int \frac{y_2 m_0(z_j)}{t+m_0(z_j)} dH(t) \right)}{1 - \int \frac{y_2 m_0^2(z_j)}{(t+m_0(z_j))^2} dH(t)}} (z_{j+1} - z_j) \right] \\ &\quad - \frac{\kappa-1}{4\pi} \sum_{j=0}^{2m_1+2m_2+3} \Im \left[ f'(z_j) \log \left( 1 - y_2 \int \frac{m_0^2(z_j) dH(t)}{(t+m_0(z_j))^2} \right) (z_{j+1} - z_j) \right] \\ &\quad + \frac{\beta_y y_2}{4\pi} \sum_{j=0}^{2m_1+2m_2+3} \Im \left[ f'(z_j) \left( \int \frac{m_0^2(z_j) dH(t)}{(t+m_0(z_j))^2} \right) (z_{j+1} - z_j) \right], \quad z_j \in \mathcal{A}_1 \end{aligned} \quad (4.10)$$

$$\begin{aligned} &\text{Cov}(X_{f_i}, X_{f_j}) \\ &\approx -\frac{\kappa}{4\pi^2} \sum_{j,k=0}^{2m_1+2m_2+3} \Re [f'_i(z_j^1) f'_j(z_k^2) \log(m_0(z_j^1) - m_0(z_k^2)) (z_{j+1}^1 - z_j^1) (z_{k+1}^2 - z_k^2)] \\ &\quad - \frac{\beta_y y_2}{4\pi^2} \sum_{j,k=0}^{2m_1+2m_2+3} \Re \left[ f'_i(z_j^1) f'_j(z_k^2) \left[ \int \frac{t^2 dH(t)}{(t+m_0(z_j^1))(t+m_0(z_k^2))} \right] (z_{j+1}^1 - z_j^1) (z_{k+1}^2 - z_k^2) \right] \\ &\quad - \frac{\beta_x y_1}{4\pi^2} \sum_{j,k=0}^{2m_1+2m_2+3} \Re [f'_i(z_j^1) f'_j(z_k^2) \cdot [z_j^1 z_k^2 \underline{m}(z_j^1) \underline{m}(z_k^2) \cdot h_{v_1}(z_j^1, z_k^2)] (z_{j+1}^1 - z_j^1) (z_{k+1}^2 - z_k^2)], \\ &\quad z_j^1 \in \mathcal{A}_1, z_k^2 \in \mathcal{A}_2. \end{aligned} \quad (4.11)$$

## 5 Applications to high-dimensional statistical analysis

In this section, we discuss two applications of the theory developed in the paper to two high-dimensional statistical problems.

### 5.1 Power function for testing the equality of two high-dimensional covariance matrices

First we consider the two-sample test of the hypothesis that two high-dimensional covariance matrices are equal, i.e.

$$H_0 : \Sigma_1 = \Sigma_2 \quad v.s. \quad H_1 : \Sigma_1 \neq \Sigma_2 . \quad (5.1)$$

By Bai et al. [7], the likelihood ratio test statistic for (5.1) is

$$T_N = \sum_{i=1}^p \log(y_{n_1} + y_{n_2} \lambda_i) - \sum_{i=1}^p \frac{y_{n_2}}{y_{n_1} + y_{n_2}} \log \lambda_i - \log(y_{n_1} + y_{n_2})$$

where  $\lambda_i$ 's are eigenvalues of a Fisher matrix  $\mathbf{AB}^{-1}$  where

$$\mathbf{A} = \frac{1}{n_1 - 1} \sum_{k=1}^{n_1} \Sigma_1^{\frac{1}{2}} (\mathbf{X}_{\cdot k} - \bar{\mathbf{X}})(\mathbf{X}_{\cdot k} - \bar{\mathbf{X}})^T \Sigma_1^{\frac{1}{2}}, \quad \mathbf{B} = \frac{1}{n_2 - 1} \sum_{k=1}^{n_2} \Sigma_2^{\frac{1}{2}} (\mathbf{Y}_{\cdot k} - \bar{\mathbf{Y}})(\mathbf{Y}_{\cdot k} - \bar{\mathbf{Y}})^T \Sigma_2^{\frac{1}{2}}.$$

As mentioned in Introduction, this two-sample test has been widely discussed in the high-dimensional context by several authors, see e.g. Li and Chen [14] and Schott [24] which used different test statistics. Under  $H_0$  and as  $\mathbf{n} \rightarrow \infty$ , we have

$$\widetilde{T}_N = v(f)^{-\frac{1}{2}} [T_N - p \cdot F_{y_{N_1}, y_{N_2}}(f) - m(f)] \xrightarrow{H_0} N(0, 1). \quad (5.2)$$

where  $N_i = n_i - 1$ ,  $y_{n_i} = \frac{p}{n_i}$ ,  $y_{N_i} = \frac{p}{N_i}$  for  $i = 1, 2$ , and  $F_{y_{N_1}, y_{N_2}}(f)$ ,  $m(f)$  and  $v(f)$  are given in (4.5)-(4.7) of [7] with  $f(x) = \log(y_1 + y_2 x) - \frac{y_2}{y_1 + y_2} \log x$ . The critical region of asymptotic size  $\alpha = 0.05$  is

$$T_N > 1.64v(f)^{\frac{1}{2}} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f).$$

By Theorem 3.1 in this paper, under  $H_1$  we have

$$v^1(f)^{-\frac{1}{2}} [T_N - p \cdot F_{y_{N_1}, y_{N_2}}^1(f) - m^1(f)] \xrightarrow{H_1} N(0, 1),$$

where  $m^1(f)$  and  $v^1(f)$  can be approximated by (4.10) and (4.11), and  $F_{y_{N_1}, y_{N_2}}^1(f)$  by

$$F_{y_{N_1}, y_{N_2}}^1(f) = \int_{c_1}^{c_2} f(x) u_{\mathbf{y}}(x) dx \approx \frac{c_2 - c_1}{10^4} \sum_{j=1}^{10^4} f(x_j) u_{\mathbf{y}}(x_j), \quad x_j = c_1 + \frac{(c_2 - c_1)j}{10^4}$$

and  $u_{\mathbf{y}}(x_j)$  is computed by (4.9). Since

$$\begin{aligned} T_N &\geq 1.64v(f)^{\frac{1}{2}} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) \\ \Leftrightarrow v^1(f)^{-\frac{1}{2}} &\left[ T_N - p \cdot F_{y_{N_1}, y_{N_2}}^1(f) - m^1(f) \right] \\ &\geq v^1(f)^{-\frac{1}{2}} \left[ 1.64v(f)^{\frac{1}{2}} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) - p \cdot F_{y_{N_1}, y_{N_2}}^1(f) - m^1(f) \right], \end{aligned}$$

the power function of the test is

$$1 - \Phi \left( v^1(f)^{-\frac{1}{2}} \left[ 1.64v(f)^{\frac{1}{2}} + p \cdot F_{y_{N_1}, y_{N_2}}(f) + m(f) - p \cdot F_{y_{N_1}, y_{N_2}}^1(f) - m^1(f) \right] \right),$$

where  $\Phi(\cdot)$  is the standardized normal distribution function.

## 5.2 Confidence interval of $\theta$ in $\mathbf{T}_p(\theta)$

As a second application, we consider  $\mathbf{T}_p = \mathbf{T}_p(\theta)$ , that is,  $\mathbf{T}_p$  is determined by parameter  $\theta$  which takes values in an interval  $[a, b]$ . We are interested in the confidence interval for the parameter  $\theta$ . Then using the fact

$$v^\theta(f)^{-\frac{1}{2}} \left[ T_N - p \cdot F_{y_{N_1}, y_{N_2}}^\theta(f) - m^\theta(f) \right] \xrightarrow{H_1} N(0, 1),$$

we will give a method to determine the confidence interval of parameter  $\theta$ .

First cut  $[a, b]$  as  $\mathcal{A}_3 = \{\theta_j = a + \frac{(b-a)j}{m}, j = 0, \dots, m\}$  where  $m$  is a large integer, e.g.  $10^4$ . Giving  $\theta_j$ , that is,  $\mathbf{T}_p = \mathbf{T}_p(\theta_j)$  and using Algorithms 1-2, we obtain  $m^{\theta_j}(f) = EX_f$ ,  $v^{\theta_j}(f) = \text{Cov}(X_f, X_f)$  and  $F_{y_{N_1}, y_{N_2}}^{\theta_j}(f)$ ,  $j = 0, \dots, m$ . Then the confidence interval of  $\theta$  is  $[\theta_L, \theta_U]$  where

$$\theta_L = \min \left\{ \theta_j : v^{\theta_j}(f)^{-\frac{1}{2}} \left( T_N - p \cdot F_{y_{N_1}, y_{N_2}}^{\theta_j}(f) - m^{\theta_j}(f) \right) \leq 1.64 \right\},$$

and

$$\theta_U = \max \left\{ \theta_j : v^{\theta_j}(f)^{-\frac{1}{2}} \left( T_N - p \cdot F_{y_{N_1}, y_{N_2}}^{\theta_j}(f) - m^{\theta_j}(f) \right) \leq 1.64 \right\}.$$



## 6 Concluding remarks

In this paper, we have considered a general Fisher matrix  $\mathbf{F}$  where the (high-dimensional) population covariance matrices  $\mathbf{\Sigma}_1$  and  $\mathbf{\Sigma}_2$  can be arbitrary and not necessarily equal. First the limiting distribution of its eigenvalues has been found. Next and more importantly, we establish a CLT for its linear spectral statistics. This CLT is unavoidable in any two-sample statistical analysis with high-dimensional data. Besides, this CLT extends and covers the CLT of Zheng [27] which is related to standard Fisher matrices.

An important and unsolved issue on the developed theory is about the evaluation of the limiting mean and covariance function in the CLT. These functions have a very complex structure depending on non-trivial contour integrals. In the special case where the matrices  $\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1$  are diagonal, we have proposed some simplification though the obtained results are still complex. In Section 4, we have devised some numerical procedures to approximate numerically these asymptotic parameters. The advantage of these procedures is that they depend on the observed data only. However, the accuracy of these procedure is currently unknown. A precise analysis of these procedures or finding other more accurate procedures for the approximation are certainly a valuable and challenging question in future research.

## A Appendix: Proofs

### A.1 Proof of Theorem 2.1

Let

$$s_{n_2}(z) = \int_0^\infty \frac{1}{t-z} dG_{n_2}(t), \quad s_{y_2}(z) = \int_0^\infty \frac{1}{t-z} dG_{y_2}(t),$$

be the Stieltjes transforms of the ESD and LSD  $G_{y_2}(t)$  of random matrix  $(\mathbf{T}_p^{\frac{1}{2}})^* \mathbf{S}_2^{-1} \mathbf{T}_p^{\frac{1}{2}}$ , respectively. Let

$$m_{y_2}(z) = \int_0^\infty \frac{t}{1-tz} dG_{y_2}(t), \tag{A.1}$$

which is the Stieltjes transform of the image measure of  $G_{y_2}$  by the reciprocal transformation  $\lambda \mapsto 1/\lambda$  on  $(0, \infty)$ . It is easily checked that the Stieltjes transforms are related as in

$$m_{y_2}(z) = -\frac{1}{z} - \frac{1}{z^2} s_{y_2}(1/z). \tag{A.2}$$

Similarly, consider the image measure and the associated Stieltjes transform

$$m_{n_2}(z) = -\frac{1}{z} - \frac{1}{z^2} s_{n_2}(1/z), \quad m_{y_{n_2}}(z) = -\frac{1}{z} - \frac{1}{z^2} s_{y_{n_2}}(1/z). \tag{A.3}$$

Let

$$\underline{m}_{y_2}(z) = -\frac{1-y_2}{z} + y_2 m_{y_2}(z), \quad (\text{A.4})$$

then by Theorem 2.1 of Zheng, Bai and Yao [28], we have

$$z = -\frac{1}{\underline{m}_{y_2}(z)} + y_2 \int \frac{dH(t)}{t + \underline{m}_{y_2}(z)}, \quad (\text{A.5})$$

where  $H(t)$  is the LSD of  $\mathbf{T}_p$ . In fact, we have

$$\underline{m}_{y_2}(z) = -\frac{1}{z} - \frac{y_2}{z^2} s_{y_2}(1/z) \quad \text{or} \quad -\frac{1}{z} \underline{m}_{y_2}\left(\frac{1}{z}\right) = 1 + y_2 z s_{y_2}(z).$$

By Silverstein and Choi [20], we have

$$z = -\frac{1}{\underline{m}(z)} + y_1 \int \frac{tdG_{y_2}(t)}{1 + t\underline{m}(z)} = -\frac{1}{\underline{m}(z)} + y_1 m_{y_2}(-\underline{m}(z)). \quad (\text{A.6})$$

So by (A.4) the above equation reduces to

$$z = -\frac{h^2}{\underline{m}(z) \cdot y_2} + \frac{y_1}{y_2} m_{y_2}(-\underline{m}(z)). \quad (\text{A.7})$$

where  $h^2 = y_1 + y_2 - y_1 y_2$ . Write  $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z)) = \frac{1-y_2}{\underline{m}(z)} + y_2 \int \frac{tdG_{y_2}(t)}{1+t\underline{m}(z)}$ . Replacing  $z$  by  $-\underline{m}(z)$ , Eq. (A.5) becomes

$$-\underline{m}(z) = -\frac{1}{m_0(z)} + y_2 \int \frac{dH(t)}{t + m_0(z)}. \quad (\text{A.8})$$

Therefore, Eq. (A.7) reduces to

$$z = \frac{h^2 m_0(z)}{y_2(-1 + y_2 \int \frac{m_0(z)dH(t)}{t+m_0(z)})} + \frac{y_1}{y_2} m_0(z). \quad (\text{A.9})$$

The proof of Theorem 2.1 is then completed. ■

## A.2 Some useful identities

**Lemma A.1** Let  $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z))$  where  $\underline{m}(z)$  is the solution of (A.6), then we have the following identities

$$1 - y_1 \int \frac{\underline{m}^2(z)x^2 dG_{y_2}(x)}{(1 + x\underline{m}(z))^2} = \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)}, \quad (\text{A.10})$$

$$\left[ \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)} \right) \right]' = \frac{-2y_1 \int \frac{\underline{m}^3(z)(z)x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^3}}{\left[1 - y_1 \int \frac{\underline{m}^2(z)x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^2}\right]^2}, \quad (\text{A.11})$$

$$\left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)} \right)' = \frac{-2y_1 \int \frac{\underline{m}^3(z)(z)x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^3}}{1 - y_1 \int \frac{\underline{m}^2(z)x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^2}}, \quad (\text{A.12})$$

$$\left[ \log \left( 1 - y_2 \int \frac{m_0^2 dH(t)}{(t+m_0)^2} \right) \right]' = \frac{2\underline{m}'(z)y_2 \int \frac{tm_0^3 dH(t)}{(t+m_0)^3}}{\left(1 - y_2 \int \frac{m_0^2 dH(t)}{(t+m_0)^2}\right)^2}, \quad (\text{A.13})$$

$$m_0(z) = \frac{1}{\underline{m}(z)} \left( 1 - \frac{y_2}{\underline{m}(z)} s_{y_2} \left( -\frac{1}{\underline{m}(z)} \right) \right), \quad m_0' = \frac{-\underline{m}'m_0^2}{1 - y_2 \int \frac{m_0^2 dH(t)}{(t+m_0)^2}} \quad (\text{A.14})$$

$$1 - y_1 \int \frac{(\underline{m}(z))^2 x^2 dG_{y_2}(x)}{(1 + x\underline{m}(z))^2} = \frac{(\underline{m}(z))^2}{\underline{m}'(z)}, \quad (\text{A.15})$$

$$\left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2} \right)' = 2\underline{m}'(z) \frac{y_2 \int \frac{m_0^3(z)t}{(t+m_0(z))^3} dH(t)}{1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2}}, \quad (\text{A.16})$$

where  $m_0'(z) = \frac{d}{dz} m_0(z)$  and  $\underline{m}'(z) = \frac{d}{dz} \underline{m}(z)$ .

**Proof.** By (A.1), we have  $m_{y_2}'(z) = \int_0^\infty \frac{x^2 dG_{y_2}(x)}{(1-xz)^2}$  where ' denotes derivative. So by (A.4) we have

$$\int \frac{x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^2} = m_{y_2}'(-\underline{m}(z)) = -\frac{1-y_2}{y_2} \cdot \frac{1}{(\underline{m}(z))^2} + \frac{1}{y_2} \cdot \underline{m}_{y_2}'(-\underline{m}(z)). \quad (\text{A.17})$$

where  $\underline{m}_{y_2}'(-\underline{m}(z)) = \frac{d}{d\xi} \underline{m}_{y_2}(\xi)|_{\xi=-\underline{m}(z)}$  instead of  $\frac{d}{dz} \underline{m}_{y_2}(-\underline{m}(z))$ . By (A.17), we have

$$1 - y_1 \int \frac{(\underline{m}(z))^2 x^2 dG_{y_2}(x)}{(1+x\underline{m}(z))^2} = \frac{h^2}{y_2} - \frac{y_1(\underline{m}(z))^2 \underline{m}_{y_2}'(-\underline{m}(z))}{y_2}. \quad (\text{A.18})$$

Differentiating both sides of (A.5) and then replacing  $z$  by  $-\underline{m}$ , we obtain

$$1 = \left( \frac{1}{m_0^2} - y_2 \int \frac{dH(t)}{(t+m_0)^2} \right) m_{y_2}'(-\underline{m}). \quad (\text{A.19})$$

This equation, together with (A.8), (A.18) and (A.19) imply that

$$1 - y_1 \int \frac{\underline{m}^2(z)x^2 dG_{y_2}(x)}{(1 + x\underline{m}(z))^2} = \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)}. \quad (\text{A.20})$$

Differentiating both sides of (A.7) with respect to  $z$ , we obtain

$$1 = \frac{h^2}{y_2(\underline{m}(z))^2} \underline{m}'(z) - \frac{y_1}{y_2} \underline{m}'_{y_2}(-\underline{m}(z)) \underline{m}'(z).$$

This implies that

$$\underline{m}'(z) = \frac{y_2(\underline{m}(z))^2}{h^2 - y_1(\underline{m}(z))^2 \underline{m}'_{y_2}(-\underline{m}(z))},$$

or equivalently

$$y_1(\underline{m}(z))^2 \underline{m}'_{y_2}(-\underline{m}(z)) = h^2 - \frac{y_2(\underline{m}(z))^2}{\underline{m}'(z)}. \quad (\text{A.21})$$

So by (A.18) and (A.21), we have

$$1 - y_1 \int \frac{(\underline{m}(z))^2 x^2 dG_{y_2}(x)}{(1 + x\underline{m}(z))^2} = \frac{(\underline{m}(z))^2}{\underline{m}'(z)}. \quad (\text{A.22})$$

Differentiating both sides of (A.8), we have

$$m'_0 = \frac{-\underline{m}' m_0^2}{1 - y_2 \int \frac{m_0^2 dH(t)}{(t+m_0)^2}}. \quad (\text{A.23})$$

So we have

$$\left(1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t + m_0(z))^2}\right)' = 2\underline{m}'(z) \frac{y_2 \int \frac{m_0^3(z)t}{(t+m_0(z))^3} dH(t)}{1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2}}.$$

So by (A.20) and (A.22), we obtain (A.11). By (A.23), we have the conclusion (A.13). By (A.2) and (A.4), we have

$$m_0(z) = \frac{1}{\underline{m}(z)} \left(1 - \frac{y_2}{\underline{m}(z)} s_{y_2} \left(-\frac{1}{\underline{m}(z)}\right)\right).$$

The proof of the lemma is completed. ■

In the sequel, for brevity,  $s_{y_2}(z)$  will denoted as  $s(z)$  if no confusion would be possible.

### A.3 Proof of Theorem 3.1

#### A.3.1 Deriving CLT of general Fisher matrix

Following the same techniques of truncation and normalisation given in Bai and Silverstein [5] (see lines -9 to -6 from the bottom of Page 559), we may assume the following additional assumptions:

- $|X_{jk}| < \eta_p \sqrt{p}$ ,  $|Y_{jk}| < \eta_p \sqrt{p}$ , for some  $\eta_p \rightarrow 0$ , as  $p \rightarrow \infty$ ,
- $EX_{jk} = 0$ ,  $EY_{jk} = 0$  and  $E|X_{jk}|^2 = 1$ ,  $E|Y_{jk}|^2 = 1$ ;
- $E|X_{jk}|^4 = 1 + \kappa + \beta_x + o(1)$  and  $E|Y_{jk}|^4 = 1 + \kappa + \beta_y + o(1)$ ;
- For the complex case,  $EX_{jk}^2 = o(n_1^{-1})$  and  $EY_{jk}^2 = o(n_2^{-1})$ .

We have

$$n_1 [\underline{m}_{\mathbf{n}}(z) - \underline{m}_{\mathbf{y}_{\mathbf{n}}}(z)] = n_1 [\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)] + n_1 [\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y}_{\mathbf{n}}}(z)]$$

where  $\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)$  and  $\underline{m}_{\mathbf{y}_{\mathbf{n}}}(z)$  are the unique roots with imaginary parts having the same signs as that of  $z$  to the following equations by (2.12)

$$z = -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + y_{n_1} \cdot \int \frac{tdG_{n_2}(t)}{1 + t\underline{m}^{\{y_{n_1}, G_{n_2}\}}} \quad \text{and} \quad z = -\frac{1}{\underline{m}_{\mathbf{y}_{\mathbf{n}}}} + y_{n_1} \cdot \int \frac{tdG_{y_{n_2}}(t)}{1 + t\underline{m}_{\mathbf{y}_{\mathbf{n}}}}.$$

The proof follows two steps and we unify the real and complex cases with the indicator notation  $\kappa$ .

**Step 1.** Consider the conditional distribution of

$$n_1 [\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)]. \quad (\text{A.24})$$

given  $\mathcal{S}_2 = \{\text{all } \mathbf{S}_2\}$ . In the proof of Theorem 2.1, we have proved that  $G_{n_2}$  converges to  $G_{y_2}$ . Using Lemma 1.1 of Bai and Silverstein (2004), we conclude that the conditional distribution of

$$n_1 [\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)] = p [m_{\mathbf{n}}(z) - m^{\{y_{n_1}, G_{n_2}\}}(z)]$$

given  $\mathcal{S}_2$  converges to a Gaussian process  $M_1(z)$  on the contour  $\mathcal{C}$  enclosing the support  $[a, b]$  of the LSD  $U_{\mathbf{y}}$  of Fisher matrix. Moreover, its mean function equals

$$\begin{aligned} E(M_1(z)|\mathcal{S}_2) &= (\kappa - 1) \cdot \frac{y_1 \int \underline{m}(z)^3 x^2 [1 + x\underline{m}(z)]^{-3} dG_{y_2}(x)}{\left[1 - y_1 \int \underline{m}^2(z) x^2 (1 + x\underline{m}(z))^{-2} dG_{y_2}(x)\right]^2} \\ &\quad + \beta_x \cdot \frac{h_{m1}(z)}{\left[y_1 z^2 \underline{m}^3(z)\right]^{-1} \cdot \left[1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{\{1 + x\underline{m}(z)\}^2} dG_{y_2}(x)\right]}, \quad (\text{A.25}) \end{aligned}$$

where we used the fact that the limit (3.4) exists for  $z \in \mathcal{C}$  when  $\beta_x \neq 0$  and in this case, the mean function has then an additional term

$$\frac{\beta_x}{p} \sum_{i=1}^p \frac{\mathbb{E} \left[ \mathbf{e}_i' \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{D}_1^{-1} \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \} \mathbf{e}_i \cdot \mathbf{e}_i' \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{D}_1^{-1} \left( \underline{m}(z) \{ \mathbf{T}_p^{\frac{1}{2}} \}^* \mathbf{S}_2^{-1} \mathbf{T}_p^{\frac{1}{2}} + \mathbf{I}_p \right) \{ \mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}} \} \mathbf{e}_i \right]}{[y_1 z^2 \underline{m}^3(z)]^{-1} \cdot \left\{ 1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{[1+x\underline{m}(z)]^2} dG_{y_2}(x) \right\}}.$$

The last expression is obtained by replacing  $\mathbf{S}^{-1/2}$  by  $\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}}$  in (6.40) of Zheng (2012). The conditional covariance function of the process  $M_1(z)$  equals

$$\begin{aligned} \text{Cov}(M_1(z_1), M_1(z_2) | \mathcal{S}_2) &= \kappa \cdot \left( \frac{\underline{m}'(z_1) \cdot \underline{m}'(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right) \\ &\quad + \beta_x y_1 \cdot \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \partial z_2}, \end{aligned} \quad (\text{A.26})$$

where we used the fact that the limit (3.5) exists for  $z \in \mathcal{C}$  when  $\beta_x \neq 0$  and in this case, the covariance function has then an additional term obtained by replacing  $\mathbf{S}^{-1/2}$  by  $\mathbf{S}_2^{-\frac{1}{2}} \mathbf{T}_p^{\frac{1}{2}}$  in (6.41) of Zheng (2012).

It is remarkable fact that these limiting functions are independent of the conditioning  $\mathcal{S}_2$ , which shows that the limiting process  $M_1(z)$  is independent of the limit of the second part below.

Step 2. Now, we consider the limiting process of

$$n_1 [\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y}_n}(z)] = p [m^{\{y_{n_1}, G_{n_2}\}}(z) - m_{\mathbf{y}_n}(z)]. \quad (\text{A.27})$$

By (A.1), we have

$$z = -\frac{1}{\underline{m}_{\mathbf{y}_n}} + y_{n_1} \int \frac{t}{1 + t \cdot \underline{m}_{\mathbf{y}_n}} dG_{y_{n_2}}(t) = -\frac{1}{\underline{m}_{\mathbf{y}_n}} + y_{n_1} \cdot m_{y_{n_2}}(-\underline{m}_{\mathbf{y}_n}(z)). \quad (\text{A.28})$$

On the other hand,  $\underline{m}^{\{y_{n_1}, G_{n_2}\}}$  is the solution to the equation

$$z = -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + y_{n_1} \int \frac{t \cdot dG_{n_2}(t)}{1 + t \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}}} ,$$

and

$$\begin{aligned} z &= -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + y_{n_1} \int \frac{t dG_{n_2}(t)}{1 + t \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}}} \\ &= -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + y_{n_1} \int \left\{ \frac{t dG_{n_2}(t)}{1 + t \underline{m}^{\{y_{n_1}, G_{n_2}\}}} - \frac{t dG_{n_2}(t)}{1 + t \underline{m}_{\mathbf{y}_n}} \right\} + y_{n_1} \int \frac{t dG_{n_2}(t)}{1 + t \underline{m}_{\mathbf{y}_n}}, \end{aligned} \quad (\text{A.29})$$

where

$$\int \frac{t}{1 + t \cdot \underline{m}_{\mathbf{y}_n}(z)} dG_{n_2}(t) = m_{n_2}(-\underline{m}_{\mathbf{y}_n}(z)).$$

Taking the difference of (A.28) and (A.29) yields

$$\begin{aligned}
0 &= -\frac{1}{\underline{m}^{\{y_{n_1}, G_{n_2}\}}} + \frac{1}{\underline{m}_{\mathbf{y}_n}} y_{n_1} \int \left\{ \frac{tdG_{n_2}(t)}{1+t\underline{m}^{\{y_{n_1}, G_{n_2}\}}} - \frac{tdG_{n_2}(t)}{1+t\underline{m}_{\mathbf{y}_n}} \right\} \\
&\quad + y_{n_1} \cdot \int \frac{t \cdot dG_{n_2}(t)}{1+t \cdot \underline{m}_{y_n}} - y_{n_1} \cdot \int \frac{t}{1+t \cdot \underline{m}_{\mathbf{y}_n}} dG_{y_{n_2}}(t)
\end{aligned}$$

That is,

$$\begin{aligned}
0 &= \frac{\underline{m}^{\{y_{n_1}, G_{n_2}\}} - \underline{m}_{\mathbf{y}_n}}{\underline{m}_{\mathbf{y}_n} \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}}} - y_{n_1} \int \frac{(\underline{m}^{\{y_{n_1}, G_{n_2}\}} - \underline{m}_{\mathbf{y}_n})t^2 dG_{n_2}(t)}{(1+t\underline{m}^{\{y_{n_1}, G_{n_2}\}})(1+t\underline{m}_{\mathbf{y}_n})} \\
&\quad + y_{n_1} \} m_{n_2}(-\underline{m}_{\mathbf{y}_n}) - m_{y_{n_2}}(-\underline{m}_{\mathbf{y}_n}) \} .
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&n_1 \cdot [\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y}_n}(z)] \\
&= -y_{n_1} \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}} \underline{m}_{\mathbf{y}_n} \cdot \frac{n_1 [m_{n_2}(-\underline{m}_{\mathbf{y}_n}) - m_{y_{n_2}}(-\underline{m}_{\mathbf{y}_n})]}{1 - y_{n_1} \cdot \int \frac{\underline{m}_{\mathbf{y}_n} \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}} t^2 dG_{n_2}(t)}{(1+t\underline{m}_{\mathbf{y}_n}) \cdot (1+t\underline{m}^{\{y_{n_1}, G_{n_2}\}})} \\
&= -\underline{m}^{\{y_{n_1}, G_{n_2}\}} \underline{m}_{\mathbf{y}_n} \cdot \frac{p [m_{n_2}(-\underline{m}_{\mathbf{y}_n}) - m_{y_{n_2}}(-\underline{m}_{\mathbf{y}_n})]}{1 - y_{n_1} \cdot \int \frac{\underline{m}_{\mathbf{y}_n} \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}} t^2 dG_{n_2}(t)}{(1+t\underline{m}_{\mathbf{y}_n}) \cdot (1+t\underline{m}^{\{y_{n_1}, G_{n_2}\}})} . \tag{A.30}
\end{aligned}$$

We then consider the limiting process of

$$p [m_{n_2}(-\underline{m}_{\mathbf{y}_n}(z)) - m_{y_{n_2}}(-\underline{m}_{\mathbf{y}_n}(z))] = -\frac{p}{(\underline{m}_{\mathbf{y}_n}(z))^2} \left[ s_{n_2} \left( \frac{-1}{\underline{m}_{\mathbf{y}_n}(z)} \right) - s_{y_{n_2}} \left( \frac{-1}{\underline{m}_{\mathbf{y}_n}(z)} \right) \right]$$

by (A.3). Noticing that for any  $z \in \mathbb{C} \setminus \mathbb{R}$ ,  $\underline{m}_{\mathbf{y}_n}(z) \rightarrow \underline{m}(z)$ , the limiting distribution of

$$-\frac{p}{z^2} \left[ s_{n_2} \left( \frac{-1}{\underline{m}_{\mathbf{y}_n}(z)} \right) - s_{y_{n_2}} \left( \frac{-1}{\underline{m}_{\mathbf{y}_n}(z)} \right) \right]$$

is the same as that of

$$-\frac{p}{(\underline{m}_{\mathbf{y}_n}(z))^2} \left[ s_{n_2} \left( \frac{-1}{\underline{m}(z)} \right) - s_{y_{n_2}} \left( \frac{-1}{\underline{m}(z)} \right) \right].$$

From now on, we use the notation  $g(z) = -1/\underline{m}(z)$ . By Theorem 2.2 of Zheng, Bai and Yao (2013), we conclude that

$$-pg^2(z) [s_{n_2}(g(z)) - s_{y_{n_2}}(g(z))] ,$$

converges weakly to a Gaussian process  $M_2(\cdot)$  on  $z \in \mathcal{C}$  with mean function

$$\mathbb{E}(M_2(z)) = (\kappa - 1) \cdot \frac{y_2 \int \frac{t[1+y_2g(z)s(g(z))]^3 dH(t)}{[-t\underline{m}(z)-1-y_2g(z)s(g(z))]^3}}{\left(1 - y_2 \int \frac{[1+y_2g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z)-1-y_2g(z)s(g(z))]^2}\right)^2} \quad (\text{A.31})$$

$$+ \frac{\beta_y y_2 [1 + y_2 g(z) s(g(z))]^3 h_M(g(z))}{1 - y_2 \int \frac{[1+y_2g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z)-1-y_2g(z)s(g(z))]^2}}, \quad (\text{A.32})$$

and covariance function  $\text{Cov}(M_2(z_1), M_2(z_2))$  equaling

$$\begin{aligned} \kappa g^2(z_1) g^2(z_2) & \left( \frac{\frac{\partial\{g(z_1)[1+y_2g(z_1)s(g(z_1))]\}}{\partial\{-1/\underline{m}(z_1)\}} \frac{\partial\{g(z_2)[1+y_2g(z_2)s(g(z_2))]\}}{\partial\{-1/\underline{m}(z_2)\}}}{\{g(z_1)[1+y_2g(z_1)s(g(z_1))] - g(z_2)[1+y_2g(z_2)s(g(z_2))]\}^2} - \frac{1}{[g(z_1) - g(z_2)]^2} \right) \\ & + \beta_y y_2 g^2(z_1) g^2(z_2) \frac{\partial^2 [(1+y_2g(z_1)s(g(z_1)))(1+y_2g(z_2)s(g(z_2))) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} \end{aligned} \quad (\text{A.33})$$

for  $z_1, z_2 \in \mathcal{C}$ , where  $H(t)$  is the LSD of  $\mathbf{T}_p$ . Here we have used the fact that the limits  $h_M(z)$  and  $h(z_1, z_2)$  in (3.6)-(3.7) exist whenever  $\beta_y \neq 0$ . Since

$$1 - y_{n_1} \cdot \int \frac{\underline{m}_{\mathbf{y}_n}(z) \cdot \underline{m}^{\{y_{n_1}, G_{n_2}\}} t^2 dG_{n_2}(t)}{(1 + t\underline{m}_{\mathbf{y}_n}(z)) (1 + t\underline{m}^{\{y_{n_1}, G_{n_2}\}})} \longrightarrow 1 - y_1 \int \frac{t^2 \underline{m}^2(z) dG_{y_2}(t)}{[1 + t\underline{m}(z)]^2},$$

almost surely, this limit equals  $\frac{\underline{m}^2}{\underline{m}'}$  by (A.22). Then by (A.30) we have

$$n_1 \cdot [\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y}_n}(z)],$$

converges weakly to a Gaussian process

$$M_3(z) = -\underline{m}'(z) M_2(z),$$

with mean function  $\mathbb{E}(M_3(z)) = -\underline{m}'(z) \mathbb{E} M_2(z)$  and covariance functions  $\text{Cov}(M_3(z_1), M_3(z_2)) = \underline{m}'(z_1) \underline{m}'(z_2) \text{Cov}(M_2(z_1), M_2(z_2))$ . Since the limit process  $M_1(z)$  of

$$n_1 \cdot [\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)]$$

is independent of the ESD of  $S_{n_2}$ , we know that

$$\{n_1 \cdot [\underline{m}_{\mathbf{n}}(z) - \underline{m}^{\{y_{n_1}, G_{n_2}\}}(z)], \quad n_1 \cdot [\underline{m}^{\{y_{n_1}, G_{n_2}\}}(z) - \underline{m}_{\mathbf{y}_n}(z)]\}$$

converge to a two-dimensional Gaussian process  $(M_1(z), M_3(z))$  with independent components. Consequently,  $n_1 \cdot [\underline{m}_{\mathbf{n}}(z) - \underline{m}_{\mathbf{y}_n}(z)]$  converges weakly to  $M_4(z)$ , a Gaussian process



with mean function

$$E(M_4(z)) = (\kappa - 1) \cdot \frac{y_1 \int \underline{m}^3(z) x^2 [1 + x \underline{m}(z)]^{-3} dG_{y_2}(x)}{\left[1 - y_1 \int \underline{m}^2(z) x^2 (1 + x \underline{m}(z))^{-2} dG_{y_2}(x)\right]^2} \quad (\text{A.34})$$

$$+ \beta_x \cdot \frac{h_{m1}(z)}{[y_1 z^2 \underline{m}^3(z)]^{-1} \cdot \left[1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{\{1 + x \underline{m}(z)\}^2} dG_{y_2}(x)\right]} \quad (\text{A.35})$$

$$- (\kappa - 1) \underline{m}'(z) \cdot \frac{y_2 \int \frac{[1 + y_2 g(z) s(g(z))]^3 dH(t)}{[-t \underline{m}(z) - 1 - y_2 g(z) s(g(z))]}^3}{\left(1 - y_2 \int \frac{[1 + y_2 g(z) s(g(z))]^2 dH(t)}{[-t \underline{m}(z) - 1 - y_2 g(z) s(g(z))]}^2\right)^2} \quad (\text{A.36})$$

$$- \beta_y \cdot \underline{m}'(z) \frac{y_2 [1 + y_2 g(z) s(g(z))]^3 h_M(g(z))}{1 - y_2 \int \frac{[1 + y_2 g(z) s(g(z))]^2 dH(t)}{[-t \underline{m}(z) - 1 - y_2 g(z) s(g(z))]}^2}, \quad (\text{A.37})$$

and covariance function

$$\begin{aligned} & \text{Cov}(M_4(z_1), M_4(z_2)) \\ = & \kappa \cdot \left( \frac{\underline{m}'(z_1) \cdot \underline{m}'(z_2)}{(\underline{m}(z_1) - \underline{m}(z_2))^2} - \frac{1}{(z_1 - z_2)^2} \right) + \beta_x y_1 \cdot \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \partial z_2} \\ & + \kappa g'(z_1) g'(z_2) \frac{\frac{\partial \{g(z_1)[1 + y_2 g(z_1) s(g(z_1))]\}}{\partial \{-1/\underline{m}(z_1)\}} \frac{\partial \{g(z_2)[1 + y_2 g(z_2) s(g(z_2))]\}}{\partial \{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1) s(g(z_1))] - g(z_2)[1 + y_2 g(z_2) s(g(z_2))]\}^2} \\ & - \kappa g'(z_1) g'(z_2) \frac{1}{[g(z_1) - g(z_2)]^2} \\ & + \beta_y y_2 g'(z_1) g'(z_2) \frac{\partial^2 [(1 + y_2 g(z_1) s(g(z_1))) (1 + y_2 g(z_2) s(g(z_2))) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} \\ = & -\kappa \cdot \frac{1}{(z_1 - z_2)^2} + \beta_x y_1 \cdot \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \partial z_2} \quad (\text{A.38}) \end{aligned}$$

$$+ \kappa g'(z_1) g'(z_2) \frac{\frac{\partial \{g(z_1)[1 + y_2 g(z_1) s(g(z_1))]\}}{\partial \{-1/\underline{m}(z_1)\}} \frac{\partial \{g(z_2)[1 + y_2 g(z_2) s(g(z_2))]\}}{\partial \{-1/\underline{m}(z_1)\}}}{\{g(z_1)[1 + y_2 g(z_1) s(g(z_1))] - g(z_2)[1 + y_2 g(z_2) s(g(z_2))]\}^2} \quad (\text{A.39})$$

$$+ \beta_y y_2 g'(z_1) g'(z_2) \frac{\partial^2 [(1 + y_2 g(z_1) s(g(z_1))) (1 + y_2 g(z_2) s(g(z_2))) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} \quad (\text{A.40})$$

### A.3.2 Simplifying the mean expressions (A.34) to (A.37) and the covariance expressions (A.39)-(A.40)

Recall that  $m_0(z) = \underline{m}_{y_2}(-\underline{m}(z))$ . By (A.11), we have

$$(A.34) = (\kappa - 1) \cdot \frac{y_1 \int \frac{\underline{m}^3(z) x^2}{[1 + x \underline{m}(z)]^3} dG_{y_2}(x)}{\left[1 - y_1 \int \frac{\underline{m}^2(z) x^2}{(1 + x \underline{m}(z))^2} dG_{y_2}(x)\right]^2} = \frac{-(\kappa - 1)}{2} \frac{d \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - y_2 \int \frac{m_0(z)}{t + m_0(z)} dH(t)\right)^2}{1 - y_2 \int \frac{m_0^2(z)}{(t + m_0(z))^2} dH(t)} \right)}{dz}.$$

By (A.10) we have

$$(A.35) = \beta_x \cdot \frac{h_{m1}(z)}{[y_1 z^2 \underline{m}^3(z)]^{-1} \cdot \left[1 - y_1 \int \frac{x^2 \underline{m}^2(z)}{\{1+x\underline{m}(z)\}^2} dG_{y_2}(x)\right]} = \beta_x \cdot \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)}}.$$

By (A.13) and (A.14) we have

$$(A.36) = -(\kappa-1)\underline{m}'(z) \cdot \frac{y_2 \int \frac{t[1+y_2 g(z)s(g(z))]^3 dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^3}}{\left(1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2}\right)^2} = -\frac{\kappa-1}{2} \frac{d \log \left(1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2}\right)}{dz}.$$

We have

$$(A.37) = -\beta_y \cdot \underline{m}'(z) \frac{y_2 [1 + y_2 g(z)s(g(z))]^3 h_M(g(z))}{1 - y_2 \int \frac{[1+y_2 g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z)-1-y_2 g(z)s(g(z))]^2}} = -\beta_y \cdot \underline{m}'(z) \frac{y_2 \underline{m}^3(z) m_0^3(z) h_M(g(z))}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} dH(t)}.$$

By (A.14) we have

$$(A.39) = \kappa g'(z_1) g'(z_2) \frac{\frac{\partial \{g(z_1)[1+y_2 g(z_1)s(g(z_1))]\}}{\partial \{-1/\underline{m}(z_1)\}} \frac{\partial \{g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}}{\partial \{-1/\underline{m}(z_2)\}}}{\{g(z_1)[1+y_2 g(z_1)s(g(z_1))]-g(z_2)[1+y_2 g(z_2)s(g(z_2))]\}^2}} \\ = \kappa \cdot \frac{1}{(m_0(z_1) - m_0(z_2))^2} \frac{\partial m_0(z_1)}{\partial z_1} \frac{\partial m_0(z_2)}{\partial z_2},$$

and

$$(A.40) = \beta_y y_2 g'(z_1) g'(z_2) \frac{\partial^2 [\underline{m}(z_1) m_0(z_1) \underline{m}(z_2) m_0(z_2) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))}.$$

So we obtain

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (A.34) dz \\ = \frac{\kappa-1}{4\pi i} \oint_{\mathcal{C}} f_i(z) d \log \left( \frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - y_2 \int \frac{m_0}{t+m_0} dH(t)\right)^2}{1 - y_2 \int \frac{m_0^2}{(t+m_0)^2} dH(t)} \right), \quad (A.41)$$

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (A.35) dz = -\frac{\beta_x}{2\pi i} \cdot \oint \frac{y_1 z^2 \underline{m}^3(z) \cdot h_{m1}(z)}{\frac{h^2}{y_2} - \frac{y_1}{y_2} \cdot \frac{\left(1 - \int \frac{y_2 m_0}{t+m_0} dH(t)\right)^2}{1 - \int \frac{y_2 m_0^2}{(t+m_0)^2} dH(t)}} dz, \quad (A.42)$$

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (A.36) dz = \frac{\kappa-1}{4\pi i} \oint_{\mathcal{C}} f_i(z) d \log \left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t+m_0(z))^2} \right), \quad (A.43)$$

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} f_i(z) \cdot (A.37) dz = \frac{\beta_y}{2\pi i} \cdot \oint \underline{m}'(z) \frac{y_2 \underline{m}^3(z) m_0^3(z) h_M(g(z))}{1 - y_2 \int \frac{m_0^2(z)}{(t+m_0(z))^2} dH(t)} dz, \quad (A.44)$$

for the mean function, where  $h^2 = y_1 + y_2 - y_1 y_2$ , and

$$\begin{aligned} & -\frac{1}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \cdot (A.38) dz \\ & = -\frac{\beta_x y_1}{4\pi^2} \cdot \oint \oint \frac{\partial^2 [z_1 z_2 \underline{m}(z_1) \underline{m}(z_2) h_{v1}(z_1, z_2)]}{\partial z_1 \partial z_2} dz_1 dz_2, \end{aligned} \quad (A.45)$$

$$\begin{aligned} & -\frac{1}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \cdot (A.39) dz \\ & = -\frac{\kappa}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{f_i(z_1) f_j(z_2)}{(m_0(z_1) - m_0(z_2))^2} dm_0(z_1) dm_0(z_2), \end{aligned} \quad (A.46)$$

$$\begin{aligned} & -\frac{1}{4\pi^2} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} f_i(z_1) f_j(z_2) \cdot (A.40) dz = -\frac{\beta_y}{4\pi^2} \oint \oint \frac{y_2 \underline{m}'(z_1) \underline{m}'(z_2)}{\underline{m}^2(z_1) \underline{m}^2(z_2)} \\ & \times \frac{\partial^2 [\underline{m}(z_1) m_0(z_1) \underline{m}(z_2) m_0(z_2) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} dz_1 dz_2, \end{aligned} \quad (A.47)$$

for the covariance function. The respective sums lead to the mean and covariance functions of the Theorem.

#### A.4 Proof of Proposition 3.1

When the matrices  $\mathbf{T}_p$  are disgonal, we first find the limit functions  $h_M(z)$  and  $h(z_1, z_2)$ . This will lead to the simplication of the terms (A.44) and (A.47). We have

$$\left(\frac{1}{z} \mathbf{T}_p - \mathbf{S}_{2,k}\right)^{-1} = \left(\frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z) \mathbf{I}\right)^{-1} + B(z) + E\beta_{12}(z) \cdot A(z) + C(z),$$

where

$$\mathbf{A}(z) = \sum_{i \neq k} \left(\frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z) \mathbf{I}\right)^{-1} (\boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' - \frac{1}{n}) \left(\frac{1}{z} \mathbf{T}_p - \mathbf{S}_{ik}\right)^{-1},$$

$$\mathbf{B}(z) = \sum_{i \neq k} (\beta_{ik}(z) - E\beta_{12}(z)) \left(\frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z) \mathbf{I}\right)^{-1} \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' \left(\frac{1}{z} \mathbf{T}_p - \mathbf{S}_{ik}\right)^{-1},$$

$$\mathbf{C}(z) = \frac{1}{n} \cdot E\beta_{12}(z) \left(\frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z) \mathbf{I}\right)^{-1} \sum_{i \neq k} \left[ \left(\frac{1}{z} \mathbf{T}_p - \mathbf{S}_{ik}\right)^{-1} - \left(\frac{1}{z} \mathbf{T}_p - \mathbf{S}_k\right)^{-1} \right],$$

with  $\beta_{ik}(z) = \frac{1}{1 - \alpha_i'(\frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik})^{-1}\alpha_i}$ ,  $\mathbf{S}_{2,ik} = \mathbf{S}_2 - \alpha_i\alpha_i^* - \alpha_k\alpha_k^*$  and  $\alpha_i = \frac{1}{\sqrt{n_2}}\mathbf{Y}_i$ . Then we have

$$\begin{aligned}
\mathbf{e}_l'\mathbf{A}(z)\mathbf{e}_l &= \sum_{i \neq k} \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \left( \alpha_i\alpha_i' - \frac{1}{n} \right) \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{e}_l \\
&= \sum_{i \neq k} \alpha_i' \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \alpha_i \\
&\quad - \sum_{i \neq k} \frac{1}{n} \text{tr} \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \\
&= \sum_{i \neq k} \hat{\gamma}_i^1,
\end{aligned}$$

where

$$\begin{aligned}
\hat{\gamma}_i^1 &= \alpha_i' \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \alpha_i \\
&\quad - \sum_{i \neq k} \frac{1}{n} \text{tr} \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathbf{e}_l'\mathbf{A}(z)\mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \\
&= \sum_{i \neq k} \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \left( \alpha_i\alpha_i' - \frac{1}{n} \right) \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \\
&= \sum_{i \neq k} \alpha_i' \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \alpha_i \\
&\quad - \sum_{i \neq k} \frac{1}{n} \text{tr} \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \\
&= \sum_{i \neq k} \hat{\gamma}_i^2,
\end{aligned}$$

where

$$\begin{aligned}
\hat{\gamma}_i^2 &= \alpha_i' \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1} \alpha_i \\
&\quad - \sum_{i \neq k} \frac{1}{n} \text{tr} \left( \frac{1}{z}\mathbf{T}_p - \mathbf{S}_{2,ik} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z}\mathbf{T}_p - (1 + y_2zs(z))\mathbf{I} \right)^{-1} \mathbf{e}_l \mathbf{e}_l' \left( \frac{1}{z}\mathbf{T}_p - \frac{n-1}{n}E\beta_{12}(z)\mathbf{I} \right)^{-1}.
\end{aligned}$$

So we obtain

$$\begin{aligned} & \left| \mathbf{E} \mathbf{e}'_l \mathbf{A}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{A}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \\ &= \left| \mathbf{E} \sum_{i \neq k} \hat{\gamma}_i^1 \hat{\gamma}_i^2 \right|^2 = \sum_{i \neq k} \mathbf{E} |\hat{\gamma}_i^1 \hat{\gamma}_i^2|^2 \leq \sum_{i \neq k} \sqrt{\mathbf{E} |\hat{\gamma}_i^1|^4 \mathbf{E} |\hat{\gamma}_i^2|^4} \leq K \cdot \eta_n^4. \end{aligned}$$

Similarly,

$$\begin{aligned} & \left| \mathbf{E} \mathbf{e}'_l \mathbf{A}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{B}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{A}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{C}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{B}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{A}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{B}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{B}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{B}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{C}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{C}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{A}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{C}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{B}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4, \\ & \left| \mathbf{E} \mathbf{e}'_l \mathbf{C}(z) \mathbf{e}_l \cdot \mathbf{e}'_l \mathbf{C}(z) \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_l \right|^2 \leq K \cdot \eta_n^4. \end{aligned}$$

Then it is easy to obtain

$$\begin{aligned} & \frac{1}{p} \sum_{j=1}^p \mathbf{E} \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_{2,k} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_{2,k} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_j \\ &= \frac{1}{p} \sum_{j=1}^p \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E \beta_{12}(z) \mathbf{I} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - \frac{n-1}{n} E \beta_{12}(z) \mathbf{I} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_j \\ & \quad + o(1) \\ &= \frac{1}{p} \sum_{j=1}^p \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_j \\ & \quad + o(1), \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{pn_2} \sum_{j=1}^p \sum_{i=1}^{n_2} \mathbf{e}'_j \mathbf{E}_i \left( \frac{1}{z_1} \mathbf{T}_p - \mathbf{S}_{2,i} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \mathbf{E}_i \left( \frac{1}{z_2} \mathbf{T}_p - \mathbf{S}_{2,i} \right)^{-1} \mathbf{e}_j \\
&= \frac{1}{p} \sum_{j=1}^p \mathbf{e}'_j \left( \frac{1}{z_1} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z_1) \mathbf{I} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \left( \frac{1}{z_2} \mathbf{T}_p - \frac{n-1}{n} E\beta_{12}(z_2) \mathbf{I} \right)^{-1} \mathbf{e}_j + o_p(1) \\
&= \frac{1}{p} \sum_{j=1}^p \mathbf{e}'_j \left( \frac{1}{z_1} \mathbf{T}_p - (1 + y_2 z_1 s(z_1)) \mathbf{I} \right)^{-1} \mathbf{e}_j \cdot \mathbf{e}'_j \left( \frac{1}{z_2} \mathbf{T}_p - (1 + y_2 z_2 s(z_2)) \mathbf{I} \right)^{-1} \mathbf{e}_j + o_p(1).
\end{aligned}$$

If  $\mathbf{T}_p$  is diagonal, then

$$\mathbf{e}'_j \left( \frac{1}{z_1} \mathbf{T}_p - (1 + y_2 z_1 s(z_1)) \mathbf{I} \right)^{-1} \mathbf{e}_j = \frac{1}{\frac{\lambda_j^0}{z_1} - (1 + y_2 z_1 s(z_1))},$$

and

$$\mathbf{e}'_j \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \mathbf{e}_j = \frac{\lambda_j^0}{\left( \frac{1}{z} \lambda_j^0 - (1 + y_2 z s(z)) \right)^2},$$

where  $\lambda_j^0$ s are eigenvalues of  $\mathbf{T}_p$ . So we obtain

$$\begin{aligned}
& \frac{1}{p} \sum_{l=1}^p \left[ \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_{2k} \right)^{-1} \right]_{ll} \left[ \left( \frac{1}{z} \mathbf{T}_p - \mathbf{S}_{2k} \right)^{-1} \mathbf{T}_p \left( \frac{1}{z} \mathbf{T}_p - (1 + y_2 z s(z)) \mathbf{I} \right)^{-1} \right]_{ll} \\
&= \frac{1}{p} \sum_{j=1}^p \frac{\lambda_j^0}{\left( \frac{1}{z} \lambda_j^0 - (1 + y_2 z s(z)) \right)^3} + o(1) \\
&= \int \frac{t}{\left( \frac{t}{z} - (1 + y_2 z s(z)) \right)^3} dH_p(t) + o(1) \\
&= \int \frac{t}{\left( \frac{t}{z} - (1 + y_2 z s(z)) \right)^3} dH(t) + o(1)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{p} \sum_{j=1}^p \left[ \left( \frac{1}{z_1} \mathbf{T}_p - \mathbf{S}_{2i} \right)^{-1} \right]_{jj} \left[ \left( \frac{1}{z_2} \mathbf{T}_p - \mathbf{S}_{2i} \right)^{-1} \right]_{jj} \\
&= \frac{1}{p} \sum_{j=1}^p \frac{1}{\frac{\lambda_j^0}{z_1} - (1 + y_2 z_1 s(z_1))} \frac{1}{\frac{\lambda_j^0}{z_2} - (1 + y_2 z_2 s(z_2))} + o(1) \\
&= \int \frac{1}{\left( \frac{t}{z_1} - (1 + y_2 z_1 s(z_1)) \right) \left( \frac{t}{z_2} - (1 + y_2 z_2 s(z_2)) \right)} dH_p(t) + o(1) \\
&= \int \frac{1}{\left( \frac{t}{z_1} - (1 + y_2 z_1 s(z_1)) \right) \left( \frac{t}{z_2} - (1 + y_2 z_2 s(z_2)) \right)} dH(t) + o(1).
\end{aligned}$$

That is, we have found the limits (3.6)-(3.7) with

$$h_M(z) = \int \frac{t}{\left(\frac{t}{z} - (1 + y_2 z s(z))\right)^3} dH(t),$$

and

$$h(z_1, z_2) = \int \frac{1}{\left(\frac{t}{z_1} - (1 + y_2 z_1 s(z_1))\right) \left(\frac{t}{z_2} - (1 + y_2 z_2 s(z_2))\right)} dH(t).$$

So we have

$$\begin{aligned} h_M(g(z)) &= \int \frac{t}{(-t\underline{m}(z) - (1 + y_2 g(z)s(g(z))))^3} dH(t) \\ &= -\frac{1}{\underline{m}^3(z)} \int \frac{t}{\left(t + \frac{1}{\underline{m}(z)} (1 + y_2 g(z)s(g(z)))\right)^3} dH(t) \\ &= -\frac{1}{\underline{m}^3(z)} \int \frac{t}{(t + m_0(z))^3} dH(t) \quad (\text{by (A.14)}), \end{aligned}$$

and

$$h(g(z_1), g(z_2)) = \frac{1}{\underline{m}(z_1)\underline{m}(z_2)} \int \frac{1}{(t + m_0(z_1))(t + m_0(z_2))} dH(t).$$

Then we obtain

$$\begin{aligned} (A.37) &= -\beta_y \underline{m}'(z) \frac{y_2 [1 + y_2 g(z)s(g(z))]^3 h_M(g(z))}{1 - y_2 \int \frac{[1 + y_2 g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z) - 1 - y_2 g(z)s(g(z))]^2}} \\ &= \frac{\beta_y \underline{m}'(z) y_2 [1 + y_2 g(z)s(g(z))]^3}{1 - y_2 \int \frac{[1 + y_2 g(z)s(g(z))]^2 dH(t)}{[-t\underline{m}(z) - 1 - y_2 g(z)s(g(z))]^2}} \frac{1}{\underline{m}^3(z)} \int \frac{t}{(t + m_0(z))^3} dH(t) \\ &= -\beta_y \cdot \underline{m}'(z) \frac{y_2 \int \frac{t \underline{m}_0^3(z)}{(t + m_0(z))^3} dH(t)}{1 - y_2 \int \frac{\underline{m}_0^2(z) dH(t)}{(t + m_0(z))^2}} \\ &= \frac{\beta_y}{2} \frac{d \left( 1 - y_2 \int \frac{\underline{m}_0^2(z) dH(t)}{(t + m_0(z))^2} \right)}{dz}, \end{aligned}$$

and

$$\begin{aligned} (A.40) &= \frac{\beta_y y_2 \underline{m}'(z_1) \underline{m}'(z_2)}{\underline{m}^2(z_1) \underline{m}^2(z_2)} \frac{\partial^2 [(1 + y_2 g(z_1)s(g(z_1)))(1 + y_2 g(z_2)s(g(z_2))) h(g(z_1), g(z_2))]}{\partial(-1/\underline{m}(z_1)) \partial(-1/\underline{m}(z_2))} \\ &= \beta_y y_2 \frac{\partial^2 \int \frac{m_0(z_1) m_0(z_2)}{(t + m_0(z_1))(t + m_0(z_2))} dH(t)}{\partial z_1 \partial z_2} \\ &= \beta_y y_2 \int \frac{t^2 dH(t)}{(t + m_0(z_1))^2 (t + m_0(z_2))^2} \cdot \frac{\partial^2 m_0(z_1) m_0(z_2)}{\partial z_1 \partial z_2}. \end{aligned}$$

We have

$$(A.44) = -\frac{1}{2\pi i} \oint f(z) \cdot (A.37) dz = \frac{\beta_y}{4\pi i} \oint f(z) d \left( 1 - y_2 \int \frac{m_0^2(z) dH(t)}{(t + m_0(z))^2} \right).$$

and

$$\begin{aligned} (A.47) &= -\frac{1}{4\pi^2} \oint \oint f_i(z_1) f_j(z_2) \cdot (A.40) dz_1 dz_2 \\ &= -\frac{\beta_y y_2}{4\pi^2} \oint \oint f_i(z_1) f_j(z_2) \left[ \int \frac{t^2 dH(t)}{(t + m_0(z_1))^2 (t + m_0(z_2))^2} \right] dm_0(z_1) dm_0(z_2). \end{aligned}$$

## A.5 Proof of Lemma 4.1

Given  $z = x_z + i \cdot y_z$ , then

$$\begin{aligned} x_z + iy_z &= -\frac{h^2 m_0(z)}{y_2 \left( 1 - y_2 + y_2 \int \frac{t}{t + m_0(z)} dH(t) \right)} + \frac{y_1 m_0(z)}{y_2} \\ &= -\frac{h^2 m_0(z)}{y_2 \left( 1 - y_2 + y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0}{\lambda_j^0 + m_0(z)} \right)} + \frac{y_1 m_0(z)}{y_2} \\ &= -\frac{h^2 m_0(z)}{y_2 \left( 1 - y_2 + y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} - \mathbf{i} \cdot y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)} + \frac{y_1 m_0(z)}{y_2} \\ &= -\frac{h^2 u_0 \left( 1 - y_2 + y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right) - h^2 v_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2}}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(1 + \lambda_j^0 u_0)^2 + (\lambda_j^0)^2 v_0^2} \right)^2} + \frac{y_1 u_0}{y_2} \\ &\quad + \mathbf{i} \cdot \left[ -\frac{h^2 u_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} + h^2 v_0 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2} + \frac{y_1 v_0}{y_2} \right]. \end{aligned}$$

So we obtain

$$x_z = -\frac{h^2 u_0 \left( 1 - y_2 + y_2 \sum_{j=1}^p \frac{w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right) - h^2 v_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2}}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(1 + \lambda_j^0 u_0)^2 + (\lambda_j^0)^2 v_0^2} \right)^2} + \frac{y_1 u_0}{y_2},$$



and

$$y_z = - \frac{h^2 u_0 \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} + h^2 v_0 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)}{y_2 \left( 1 - y_2 + \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 (\lambda_j^0 + u_0)}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2 + y_2 \left( \sum_{j=1}^p \frac{y_2 w_j \lambda_j^0 v_0}{(\lambda_j^0 + u_0)^2 + v_0^2} \right)^2} + \frac{y_1 v_0}{y_2}.$$

So the proof of Lemma 4.1 is completed. ■

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